12.1 Basic Integration Formulae

Learning Objectives:

❖ To evaluate the indefinite integrals by the following seven methods

- * Making a simplifying substitution
- * Completing the square
- * Using a trigonometric identity
- * Eliminating a square root
- * Reducing an improper fraction
- * Separating a fraction
- * Multiplying by a form of 1

We evaluate an indefinite integral by finding an anti-derivative of the integrand and adding an arbitrary constant. Table 1 shows the basic forms of the integrals we have evaluated so far.

Table 1

1. $\int du = u + C$
2. $\int kdu = ku + C \ (any\ number\ k\ )$
3. $\int (du + dv) = \int du + \int dv$
4. $\int u^n du = \frac{u^{n+1}}{n+1} + C \qquad (n \neq -1)$
5. $\int \frac{du}{u} = \ln|u| + C$
6. $\int sinu\, du = -cosu + C$
7. $\int cosu\, du = sinu + C$
8. $\int sec^2 u\, du = tanu + C$
9. $\int csc^2 u\, du = -cotu + C$
10. $\int secu.tanu\, du = secu + C$
11. $\int cscu.cotu\, du = -cscu + C$
12. $\int tanu\, du = \begin{cases} -\ln|cosu| + C \\ \ \ln|secu| + C \end{cases}$
13. $\int cotu\, du = \begin{cases} \ \ln|sinu| + C \\ -\ln|cscu| + C \end{cases}$
14. $\int e^u du = e^u + C$
15. $\int a^u\, du = \frac{a^u}{\ln a} + C \qquad (a > 0,\ a \neq 1)$
16. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$
17. $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$
18. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$

1. Method of substitution:

In this section we reduce certain integrals to some standard forms by a suitable substitution.

Example:

Evaluate $\int \frac{2x-9}{\sqrt{x^2-9x+1}}dx$

Solution:

$$\int \frac{2x-9}{\sqrt{x^2-9x+1}}dx$$

$$put\ u = x^2 - 9x + 1 \Rightarrow du = 2x - 9$$

$$\int \frac{2x-9}{\sqrt{x^2-9x+1}}dx = \int \frac{du}{\sqrt{u}} = \int u^{-1/2} du$$

$$= \frac{u^{(-1/2)+1}}{(-1/2)+1} + C$$

$$where\ C\ is\ an\ arbitrary\ constant$$

$$= 2u^{1/2} + C$$

$$= 2\sqrt{x^2 - 9x + 1} + C$$

2. Method of completing the square:

In this section we reduce certain integrals to some standard forms by completing the square.

Example:

Evaluate $\int \frac{dx}{\sqrt{8x - x^2}}$

Solution:

We complete the square to write the radicand as

$$8x - x^2 = -\left(x^2 - 8x\right) = -\left(x^2 - 8x + 16 - 16\right)$$
$$= -\left(x^2 - 8x + 16\right) + 16$$
$$= 16 - (x-4)^2$$

Then

$$\int \frac{dx}{\sqrt{8x - x^2}} = \int \frac{dx}{\sqrt{16 - \left(x - 4^2\right)}}$$

$$\text{put} \quad a = 4 \text{ and } u = x - 4 \Rightarrow du = dx$$

$$\int \frac{dx}{\sqrt{8x - x^2}} = \int \frac{du}{\sqrt{a^2 - u^2}}$$
$$= \sin^{-1}\left(\frac{u}{a}\right) + C$$

where C is an arbitrary cons tan t

$$= \sin^{-1}\left(\frac{x-4}{4}\right) + C$$

3. Method of using the trigonometric identities:

In this section we reduce certain integrals to some standard forms by using the trigonometric identities.

Example:

Evaluate $\int (\sec x + \tan x)^2 \, dx$

Solution:

We expand the integrand and get

$$\begin{aligned}(\sec x + \tan x)^2 &= \sec^2 x + 2\sec x \tan x + \tan^2 x \\ &= \sec^2 x + 2\sec x \tan x + \sec^2 x - 1 \\ &= 2\sec^2 x + 2\sec x \tan x - 1\end{aligned}$$

Then

$$\begin{aligned}\int (\sec x + \tan x)^2 \, dx &= \int \left(2\sec^2 x + 2\sec x \tan x - 1\right) dx \\ &= 2\int \sec^2 x \, dx + 2\int \sec x \tan x \, dx - \int dx \\ &= 2\tan x + 2\sec x - x + C\end{aligned}$$

$$\textit{where } C \textit{ is an arbitrary constant}$$

4. Method of eliminating a square root:

In this section we reduce certain integrals to some standard forms by eliminating a square root.

Example:

Evaluate $\int_0^{\pi/4} \sqrt{1+\cos 4x}\, dx$

Solution:

We have the relation $1+\cos 2(2x) = 2\cos^2 2x$

Hence

$$\int_0^{\pi/4} \sqrt{1+\cos 4x}\, dx = \sqrt{2}\int_0^{\pi/4} |\cos 2x|\, dx \qquad \left(\because \sqrt{u^2} = |u| \right)$$

we know that $\cos 2x \geq 0$ on $[0, \pi/4]$, so $|\cos 2x| = \cos 2x$

$$\therefore \quad \int_0^{\pi/4} \sqrt{1+\cos 4x}\, dx = \sqrt{2}\int_0^{\pi/4} \cos 2x\, dx$$

$$= \sqrt{2}\left[\frac{\sin 2x}{2} \right]_0^{\pi/4}$$

$$= \sqrt{2}\left[\frac{1}{2} - 0 \right] = \frac{\sqrt{2}}{2}$$

5. Method of reducing an improper fraction:

In this section we reduce certain integrals to some standard forms by reducing an improper fraction into a proper fraction.

Example:

Evaluate $\int \frac{3x^2-7x}{3x+2}dx$

Solution:

The integrand is an improper fraction (degree of numerator greater than or equal to degree of denominator). To integrate it, we divide first, getting a quotient plus a remainder that is a proper fraction:

$$\frac{3x^2-7x}{3x+2} = x-3+\frac{6}{3x+2}$$

Therefore

$$\int \frac{3x^2-7x}{3x+2}dx = \int\left(x-3+\frac{6}{3x+2}\right)dx$$

$$= \int x\,dx - 3\int dx + 6\int\frac{dx}{3x+2}$$

$$= \frac{x^2}{2} - 3x + 6\left(\frac{\ln|3x+2|}{3}\right) + C$$

where C is an arbitrary constant

$$= \frac{x^2}{2} - 3x + 2\ln|3x+2| + C$$

6. Method of separating a fraction:

In this section we reduce certain integrals to some standard forms by separating a fraction.

Example:

Evaluate $\int \frac{3x+2}{\sqrt{1-x^2}}\,dx$

Solution:

We first separate the integrand to get

$$\int \frac{3x+2}{\sqrt{1-x^2}}\,dx = 3\int \frac{x}{\sqrt{1-x^2}}\,dx + 2\int \frac{dx}{\sqrt{1-x^2}}$$

In the first integral we substitute

$$u = 1-x^2 \Rightarrow du = -2x\,dx$$

$$3\int \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{3}{2}\int \frac{du}{\sqrt{u}}$$

$$= -\frac{3}{2}\int u^{-1/2}\,du$$

$$= -\frac{3}{2}\cdot\frac{u^{1/2}}{1/2} + C_1 = -3\sqrt{1-x^2} + C_1$$

$$\text{where } C_1 \text{ is an arbitrary constant}$$

The second integral is in the standard form

$$\therefore\ \ 2\int \frac{dx}{\sqrt{1-x^2}} = 2\sin^{-1} x + C_2$$

$$\text{where } C_2 \text{ is an arbitrary constant}$$

Combing these results and renaming $C_1 + C_2$ as C gives

$$\int \frac{3x+2}{\sqrt{1-x^2}}\,dx = -3\sqrt{1-x^2} + 2\sin^{-1} x + C$$

$$\text{where } C \text{ is an arbitrary constant}$$

7. Method of multiplying by a form of 1:

In this section we reduce certain integrals to some standard forms by multiplying by a form of 1.

Example:

Evaluate $\int \sec x \, dx$

Solution:

$$\int \sec x \, dx = \int (\sec x)(1) \, dx$$

$$= \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} \, dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx$$

$$\int \sec x \, dx = \int \frac{du}{u}$$

$$= \ln|u| + C = \ln|\sec x + \tan x| + C$$

$$\textit{where } C \textit{ is an arbitrary constant}$$

$$\boxed{\int secu \;\; du = \ln|secu + tanu| + C}$$

With cosecants and cotangents in place of secants and tangents, the method of the above example leads to a companion formula for the integral of the cosecant.

$$\boxed{\int \boldsymbol{cscu \;\; du = -\ln|cscu + cotu| + C}}$$

IP1.

$$\int \sqrt{\frac{x}{a^3 - x^3}}\ dx =$$

Solution:

Step1:

Given $\int \sqrt{\frac{x}{a^3-x^3}}\ dx = \int \frac{\sqrt{x}}{\sqrt{\left(\left(a^{\frac{3}{2}}\right)^2 - \left(x^{\frac{3}{2}}\right)^2\right)^2}}\, dx$

Now, put $x^{\frac{3}{2}} = t \Rightarrow \frac{3}{2}\sqrt{x}\, dx = dt$

Step2:

$$\int \frac{\sqrt{x}}{\sqrt{\left(\left(a^{\frac{3}{2}}\right)^2 - \left(x^{\frac{3}{2}}\right)^2\right)^2}}\, dx = \frac{2}{3}\int \frac{dt}{\sqrt{\left(a^{\frac{3}{2}}\right)^2 - t^2}}$$

$$= \frac{2}{3}\sin^{-1}\left(\frac{t}{a^{\frac{3}{2}}}\right) + C\,,$$

where C is an arbitrary constant

$$= \frac{2}{3}\sin^{-1}\left(\frac{x^{\frac{3}{2}}}{a^{\frac{3}{2}}}\right) + C$$

$$\therefore \int \sqrt{\frac{x}{a^3-x^3}}\ dx = \frac{2}{3}\sin^{-1}\left(\left(\frac{x}{a}\right)^{\frac{3}{2}}\right) + C$$

P1.

$$\int \frac{x^{49}\tan^{-1}x^{50}}{1+x^{100}}\,dx = k\left(\tan^{-1}x^{50}\right)^{2} + C \Rightarrow k =$$

Solution:

Given $\int \frac{x^{49}\tan^{-1}x^{50}}{1+x^{100}}\,dx$

Put $x^{50} = t \Rightarrow 50x^{49} = dt \Rightarrow x^{49} = \frac{1}{50}dt$

$$\int \frac{x^{49}\tan^{-1}x^{50}}{1+x^{100}}\,dx = \frac{1}{50}\int \frac{\tan^{-1}t}{1+t^2}\,dt$$

Again, put $\tan^{-1}t = z \Rightarrow \frac{dt}{1+t^2} = dz$

$$= \frac{1}{50}\int z\;dz = \frac{1}{50}\frac{z^2}{2} + C$$

where C is an arbitrary constant.

$$= \frac{(\tan^{-1}t)^2}{100} + C = \frac{(\tan^{-1}(x^{50}))^2}{100} + C$$

Hence the value of k is $\frac{1}{100}$

IP2.

$$\int \frac{dx}{3x^2+x+1} =$$

Solution:

Step1:

$$\int \frac{dx}{3x^2+x+1} = \frac{1}{3}\int \frac{dx}{x^2+\frac{x}{3}+\frac{1}{3}}$$

$$= \frac{1}{3}\int \frac{dx}{\left(x+\frac{1}{6}\right)^2-\frac{1}{36}+\frac{1}{3}}$$

$$= \frac{1}{3}\int \frac{dx}{\left(\frac{\sqrt{11}}{6}\right)^2+\left(x+\frac{1}{6}\right)^2}$$

Step2:

Put $x+\frac{1}{6}=t \Longrightarrow dx = dt$

$$\int \frac{dx}{3x^2+x+1} = \int \frac{dt}{\left(\frac{\sqrt{11}}{6}\right)^2+t^2}$$

$$= \frac{1}{3} \cdot \frac{6}{\sqrt{11}} \tan^{-1}\left(\frac{t}{\frac{\sqrt{11}}{6}}\right) + C$$

where C is an arbitrary constant

Step3:

$$\int \frac{dx}{3x^2+x+1} = \frac{2}{\sqrt{11}} \tan^{-1}\left(\frac{6\left(x+\frac{1}{6}\right)}{\sqrt{11}}\right) + C$$

$$= \frac{2}{\sqrt{11}} \tan^{-1}\left(\frac{6x+1}{\sqrt{11}}\right) + C$$

$$\therefore \int \frac{dx}{3x^2+x+1} = \frac{2}{\sqrt{11}} \tan^{-1}\left(\frac{6x+1}{\sqrt{11}}\right) + C$$

P2.

$$\int \frac{dx}{\sqrt{1 + x - x^2}} =$$

Solution:

$$\int \frac{dx}{\sqrt{1+x-x^2}} = \int \frac{dx}{\sqrt{-(x^2-x-1)}}$$

$$= \int \frac{dx}{\sqrt{-\left[\left(x-\frac{1}{2}\right)^2 - \frac{5}{4}\right]}}$$

$$= \int \frac{dx}{\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - \left(x-\frac{1}{2}\right)^2}}$$

Put $x - \frac{1}{2} = t \Rightarrow dx = dt$

$$\int \frac{dx}{\sqrt{1+x-x^2}} = \int \frac{dt}{\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - t^2}}$$

$$= \sin^{-1}\left(\frac{t}{\frac{\sqrt{5}}{2}}\right) + C$$

where C is an arbitrary constant

$$= \sin^{-1}\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{5}}{2}}\right) + C$$

$$= \sin^{-1}\left(\frac{2x-1}{\sqrt{5}}\right) + C$$

$$\therefore \int \frac{dx}{\sqrt{1+x-x^2}} = \sin^{-1}\left(\frac{2x-1}{\sqrt{5}}\right) + C$$

IP3.

$$\int \frac{\boldsymbol{sinx}}{\sqrt{\mathbf{1}+\boldsymbol{sinx}}}\, \boldsymbol{dx}$$

Solution:

Step1:

$$\int \frac{sinx}{\sqrt{1+sinx}}\, dx = \int \frac{(1+sinx)-1}{\sqrt{1+sinx}}\, dx$$

Step2:

$$\int \frac{sinx}{\sqrt{1+sinx}}\, dx$$

$$= \int \sqrt{1+sinx}\, dx - \int \frac{1}{\sqrt{1+sinx}}\, dx$$

$$= \int \sqrt{sin^2\left(\frac{x}{2}\right) + cos^2\left(\frac{x}{2}\right) + 2sin\left(\frac{x}{2}\right).cos\left(\frac{x}{2}\right)}\, dx$$

$$- \int \frac{1}{\sqrt{1+sinx}}\, dx$$

$$= \int \left[sin\left(\frac{x}{2}\right) + cos\left(\frac{x}{2}\right)\right] dx - \int \frac{1}{\left[sin\left(\frac{x}{2}\right)+cos\left(\frac{x}{2}\right)\right]}\, dx$$

$$= \left[-2cos\left(\frac{x}{2}\right) + 2sin\left(\frac{x}{2}\right)\right] - \frac{1}{\sqrt{2}}\int \frac{1}{sin\left(\frac{x}{2}\right).\frac{1}{\sqrt{2}} + cos\left(\frac{x}{2}\right).\frac{1}{\sqrt{2}}}\, dx$$

$$= 2\left[sin\left(\frac{x}{2}\right) - cos\left(\frac{x}{2}\right)\right] - \frac{1}{\sqrt{2}}\int \frac{1}{sin\left(\frac{x}{2}\right).cos\frac{\pi}{4} + cos\left(\frac{x}{2}\right) sin\frac{\pi}{4}.}\, dx$$

$$= 2\left[sin\left(\frac{x}{2}\right) - cos\left(\frac{x}{2}\right)\right] - \frac{1}{\sqrt{2}}\int \frac{1}{sin\left(\frac{x}{2}+\frac{\pi}{4}\right)}\, dx$$

$$= 2\left[sin\left(\frac{x}{2}\right) - cos\left(\frac{x}{2}\right)\right] - \frac{1}{\sqrt{2}}\int cosec\left(\frac{x}{2}+\frac{\pi}{4}\right)\, dx$$

$$= 2\left[sin\left(\frac{x}{2}\right) - cos\left(\frac{x}{2}\right)\right]$$

$$+ \frac{1}{\sqrt{2}}\ln\left|cosec\left(\frac{x}{2}+\frac{\pi}{4}\right) + cot\left(\frac{x}{2}+\frac{\pi}{4}\right)\right| + C$$

where C is an arbitrary constant

Step3:

$$\therefore \int \frac{sinx.cosx}{(sinx+cosx)}\, dx$$

$$= 2\left[sin\left(\frac{x}{2}\right) - cos\left(\frac{x}{2}\right)\right] + \frac{1}{\sqrt{2}}\ln\left|cosec\left(\frac{x}{2}+\frac{\pi}{4}\right) + cot\left(\frac{x}{2}+\frac{\pi}{4}\right)\right| + C$$

P3.

$$\int \frac{sinx.cosx}{(sinx + cosx)}\, dx$$

Solution:

$$\int \frac{sinx.cosx}{(sinx+cosx)}\,dx$$

$$= \frac{1}{2}\int \frac{2sinx.cosx}{sinx+cosx}\,dx$$

$$= \frac{1}{2}\int \frac{1+2sinx.cosx-1}{sinx+cosx}\,dx$$

$$= \frac{1}{2}\int \frac{sin^2x+cos^2x+2sinx.cosx-1}{sinx+cosx}\,dx$$

$$= \frac{1}{2}\int \frac{(sinx+cosx)^2-1}{sinx+cosx}\,dx$$

$$= \frac{1}{2}\int (sinx+cosx)\,dx - \frac{1}{2}\int \frac{1}{sinx+cosx}\,dx$$

$$= \frac{1}{2}(-cosx+sinx) - \frac{1}{2\sqrt{2}}\int \frac{1}{\frac{1}{\sqrt{2}}sinx+\frac{1}{\sqrt{2}}cosx}\,dx$$

$$= \frac{1}{2}(sinx-cosx) - \frac{1}{2\sqrt{2}}\int \frac{1}{sinx.cos\frac{\pi}{4}+cosx.sin\frac{\pi}{4}}\,dx$$

$$= \frac{1}{2}(sinx-cosx) - \frac{1}{2\sqrt{2}}\int \frac{1}{sin\left(x+\frac{\pi}{4}\right)}\,dx$$

$$= \frac{1}{2}(sinx-cosx) - \frac{1}{2\sqrt{2}}\int cosec\left(x+\frac{\pi}{4}\right)\,dx$$

$$= \frac{1}{2}(sinx-cosx)$$

$$+ \frac{1}{2\sqrt{2}}\ln\left|cosec\left(x+\frac{\pi}{4}\right)+cot\left(x+\frac{\pi}{4}\right)\right| + C$$

where C is an arbitrary constant

$$\therefore \int \frac{sinx.\,cosx}{(sinx+cosx)}\,dx$$

$$= \frac{1}{2}(sinx-cosx) + \frac{1}{2\sqrt{2}}\ln\left|cosec\left(x+\frac{\pi}{4}\right)+cot\left(x+\frac{\pi}{4}\right)\right| + C$$

IP4.

$$\int \frac{x^2 - 4x + 2}{x - 2}\, dx =$$

Solution:

Step1:

Given $\int \frac{x^2-4x+2}{x-2}\, dx$

$$\frac{x^2-4x+2}{x-2} = \frac{(x-2)^2-2}{x-2} = (x-2) - \frac{2}{(x-2)}$$

Step2:

$$\therefore \int \frac{x^2-4x+2}{x-2}\, dx = \int \left\{(x-2) - \frac{2}{(x-2)}\right\} dx$$

$$= \int x\, dx - 2\int dx - 2\int \frac{1}{(x-2)}\, dx$$

$$= \frac{x^2}{2} - 2x - 2\ln|x-2| + C$$

where C is an arbitrary constant

Step3:

$$\therefore \int \frac{x^2+x+1}{2x+3}\, dx = \frac{x^2}{2} - 2x - 2\ln|x-2| + C$$

P4.

$$\int \frac{x^2 + x + 1}{2x + 3}\, dx =$$

Solution:

Given $\int \frac{x^2+x+1}{2x+3}\, dx$

By actual division, we have

$$\frac{x^2+x+1}{2x+3} = \left(\frac{x}{2} - \frac{1}{4}\right) + \frac{7}{4(2x+3)}$$

$$\therefore \int \frac{x^2+x+1}{2x+3}\, dx = \int \left\{\left(\frac{x}{2} - \frac{1}{4}\right) + \frac{7}{4(2x+3)}\right\} dx$$

$$= \frac{1}{2}\int x\, dx - \frac{1}{4}\int dx + \frac{7}{4}\int \frac{1}{(2x+3)}\, dx$$

$$= \frac{x^2}{4} - \frac{x}{4} + \frac{7}{4}\frac{\ln|2x+3|}{2} + C$$

where C is an arbitrary constant

$$= \frac{x^2}{4} - \frac{x}{4} + \frac{7}{8}\ln|2x+3| + C$$

$$\therefore \int \frac{x^2+x+1}{2x+3}\, dx = \frac{x^2}{4} - \frac{x}{4} + \frac{7}{8}\ln|2x+3| + C$$

1) Evaluate

a) $\int \frac{(\sin^{-1} x)^2}{\sqrt{1-x^2}}\,dx$

b) $\int \sec^3 x\ dx$

c) $\int \frac{1}{1+(2x+1)^2}\,dx$

d) $\int \frac{\sin^4 x}{\cos^6 x}\,dx$

2) Evaluate

a) $\int \frac{1}{5-2x^2+4x}\,dx$

b) $\int \frac{1}{4x^2-4x-7}\,dx$

c) $\int \frac{1}{x^2+x+1}\,dx$

3)Evaluate

a) $\int \frac{1 - cos2x}{1 + cos2x}\, dx$

b) $\int \frac{cos^2 x}{1 - sinx}\, dx$

c) $\int \frac{1 + \sin^2 x}{1 + cos2x}\, dx$

d) $\int \frac{cosx + sinx}{\sqrt{1 + sin2x}}\, dx$

e) $\int \frac{1}{1 - cosx}\, dx$

4) Evaluate

a) $\int \sqrt{1 + sin\frac{x}{2}}\ dx$

b) $\int_{-\pi/4}^{\pi/4} \sqrt{1 + tan^2x}\ dx$

c) $\int_{-\pi}^{\pi} \sqrt{(1 - cos^2t)^3}\ dt$

d) $\int \sqrt{1 - sin2x}\ dx$

5) Evaluate

a) $\int \frac{x^2}{x+5}\,dx$

b) $\int \frac{x^2-5x+4}{x+2}\,dx$

c) $\int \frac{2x^2-12x-9}{2x+3}\,dx$

d) $\int \frac{3x^3-5x^2+4x-9}{x^2+2}\,dx$

6) Evaluate

a) $\int \frac{1}{cosx\ sin^2x}\,dx$

b) $\int \frac{2cos^3x + 3\ sin^3x}{cos^2x\ sin^2x}\,dx$

c) $\int \frac{x^4}{x^2+1}\,dx$

d) $\int \frac{x^2}{\sqrt{x+5}}\,dx$

7) Evaluate

$$a)\ \int \frac{x^2+2}{x^2+1}\,dx$$

$$b)\ \int \frac{x^2}{\sqrt{1+x^2}\,(1+\sqrt{1+x^2})}\,dx$$

$$c)\ \int \frac{x^5}{x^3-1}\,dx$$

12.2 Integration by parts

Learning objectives:

- To derive the formula for integration by parts.
- To execute the integration by parts by tabular integration.
  And
- To practice related problems.

Introduction:

Integration by parts is a technique for simplifying integrals of the form

$$\int f(x)g(x)dx$$

in which $f(x)$ can be differentiated repeatedly and $g(x)$ can be integrated repeatedly without difficulty.

The integral

$$\int xe^x dx$$

is such an integral because $f(x)=x$ can be differentiated twice to become zero and $g(x)=e^x$ can be integrated repeatedly without difficulty.

Integration by parts also applies to integrals like

$$\int e^x \sin x\, dx$$

in which each part of the integrand appears again after repeated differentiation or integration.

The Formula

The formula for integration by parts comes from the product rule,

$$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$$

In its differential form, the rule becomes

$$d(uv)=u\,dv+v\,du$$

which is then written as $u\,dv=d(uv)-v\,du$ and integrated to give the following formula.

$$\int u\,dv=uv-\int v\,du \quad \text{..................} (1)$$

Equation(1) is the integration-by-parts formula and it expresses one integral, $\int u\,dv$, in terms of a second integral, $\int v\,du$.

With a proper choice of u and v, the second integral may be easier to evaluate than the first. (This is the reason for the importance of the formula. When faced with an integral we cannot handle, we can replace it by one with which we might have more success)

The equivalent formula for definite integrals is

$$\int_{v_1}^{v_2} u\,dv=(u_2v_2-u_1v_1)-\int_{u_1}^{u_2} v\,du \quad \text{............} (2)$$

Example-1

Find $\int x\cos x\,dx$

Solution:

Let

$$u=x \quad and \ dv=\cos x\,dx$$

$$\Rightarrow du=dx \ and \ v=\int \cos x\,dx=\sin x$$

$$\therefore \int x\cos x\,dx=x\sin x-\int \sin x\,dx$$

$$=x\sin x+\cos x+C$$

where C is an arbitrary constant.

Example-2

Find $\int \ln x\,dx$.

Solution:

Since $\int \ln x\,dx$ can be written as $\int \ln x\cdot 1\,dx$, we use the formula $\int u\,dv=uv-\int v\,du$ with

$$u=\ln x \quad and \ dv=dx$$

$$\Rightarrow du=\frac{1}{x}dx \ and \ v=\int dx=x$$

Then

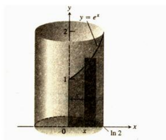
$$\int \ln x\,dx=x\ln x-\int x\cdot\frac{1}{x}dx$$

$$=x\ln x-x+C$$

where C is an arbitrary constant

Finding the volume

Find the volume of the solid generated by revolving about the y-axis the region in the first quadrant enclosed by the coordinate axes, the curve $y=e^x$ and the line $x=\ln 2$.



Solution:

Using the method of cylindrical shells, we find

$$V=\int_a^b 2\pi x f(x)\,dx$$

$$=2\pi\int_0^{\ln 2} xe^x dx$$

To evaluate the integral, we use the formula of integration-by-parts,

$$\int u\,dv=uv-\int v\,du, \text{ with}$$

$$u=x \qquad dv=e^x dx \qquad v=e^x \qquad du=dx$$

Then

$$\int_0^{\ln 2} xe^x dx=\left[xe^x\right]_0^{\ln 2}-\int_0^{\ln 2} e^x dx$$

$$=\left[\ln 2e^{\ln 2}-0\right]-\left[e^x\right]_0^{\ln 2}$$

$$=2\ln 2-(2-1)$$

$$=2\ln 2-1$$

The solid's volume is therefore

$$V=2\pi\int_0^{\ln 2} xe^x dx$$

$$=2\pi(2\ln 2-1)$$

Repeated Use

Sometimes we have to use integration by parts more than once to obtain an answer.

Example

Find $\int x^2 e^x dx$

Solution:

We use formula $\int u\,dv = uv - \int v\,du$ with

$$u = x^2 \quad and \quad dv = e^x dx$$

$$\Rightarrow du = 2x\,dx \; and \; v = \int e^x dx = e^x$$

This gives

$$\int x^2 e^x dx = x^2 e^x - 2\int x e^x dx$$

It takes a second integration by parts to find the integral on the right. We find

$$\int x e^x dx = x e^x - e^x + C'$$

where C' is an arbitrary constant

Hence $\int x^2 e^x dx = x^2 e^x - 2x e^x + 2e^x + C$

where C is an arbitrary constant

Solving for the Unknown Integral

Integrals like the one in the next example occur in electrical engineering. Their evaluation requires two integrations by parts, followed by solving for the unknown integral.

Example

Find $\int e^x \cos x\,dx$

Solution:

We first use the formula $\int u\,dv = uv - \int v\,du$ with

$$u = e^x \quad and \quad dv = \cos x\,dx$$

$$\Rightarrow du = e^x dx \; and \; v = \int \cos x \; dx = \sin x$$

Then

$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx$$

The second integral is like the first, except it has $\sin x$ in place of $\cos x$. To evaluate it, we use integration by parts with

$$u = e^x \quad and \quad dv = \sin x\,dx$$

$$\Rightarrow du = e^x dx \quad and \quad v = \int \sin x \; dx = -\cos x$$

Then

$$\int e^x \cos x\,dx = e^x \sin x - \left(-e^x \cos x - \int (-\cos x)\left(e^x dx\right)\right)$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x\,dx$$

The unknown integral now appears on both sides of the equation. Combining the two expressions gives

$$2\int e^x \cos x\,dx = e^x \sin x + e^x \cos x + C'$$

where C' is an arbitrary constant

Dividing by 2 and renaming the constant of integration gives

$$\int e^x \cos x\,dx = \frac{e^x \sin x + e^x \cos x}{2} + C$$

where C is an arbitrary constant

Tabular Integration

We have seen that integrals of the form $\int f(x)g(x)dx$, in which f can be differentiated repeatedly to become zero and g can be integrated repeatedly without difficulty, are natural candidates for integration by parts. In some cases, where there are many repetitions, the calculations can be cumbersome. There is a way to organize the calculations that saves a great deal of work. It is called ***tabular integration*** and is illustrated in the following examples.

Example

Find $\int x^2 e^x dx$ by tabular integration.

Solution:

With $f(x)=x^2$ and $g(x)=e^x$, we list

f(x) and its derivatives		**g(x) and its integrals**
x^2	(+)	e^x
$2x$	(−)	e^x
2	(+)	e^x
0		e^x.

We add the products of the functions connected by the arrows, with the middle sign changed, to obtain

$$\int x^2 e^x dx = x^2 e^x - 2xe^x + 2e^x + C$$

where C is an arbitrary constant

Example

Find $\int x^3 \sin x\, dx$ by tabular integration.

Solution:

With $f(x)=x^3$ and $g(x)=\sin x$, we list

f(x) and its derivatives		**g(x) and its integrals**
x^3	(+)	$\sin x$
$3x^2$	(−)	$-\cos x$
$6x$	(+)	$-\sin x$
6	(−)	$\cos x$
0		$\sin x$

Again we add the products of the functions connected by the arrows, with every other sign changed, to obtain

$$\int x^3 \sin x\, dx = -x^3 \cos x + 3x^2 \sin x + 6x\cos x - 6\sin x + C$$

where C is an arbitrary constant

IP1:

$$\int x\cos^2 x\, dx =$$

Solution:

Step1:

$$\int x\cos^2 x\, dx = \int \frac{x(1+\cos 2x)}{2}dx$$

$$= \frac{1}{2}\int x\, dx + \frac{1}{2}\int x\cos 2x\ dx$$

$$= \frac{x^2}{4} + \frac{1}{2}\int x\cos 2x\ dx \ \ldots\ldots\ldots\ (1)$$

Step2:

Now, to find $\int x\cos 2x\ dx$

Put $u = x$ and $dv = \cos 2x\, dx$

$\Longrightarrow du = dx$ and $v = \frac{\sin 2x}{2}$

By integration by parts, we have

$$\int u dv = uv - \int v\, dv$$

$$\int x\cos 2x\ dx = \frac{x\sin 2x}{2} - \int \frac{\sin 2x}{2}dx$$

$$= \frac{x\sin 2x}{2} + \frac{\cos 2x}{4} + C$$

where C is an arbitrary constant

Step3:

From (1), we have

$$\therefore \int x\cos^2 x\, dx = \frac{x^2}{4} + \frac{1}{2}\left[\frac{x\sin 2x}{2} - \int \frac{\sin 2x}{2}dx\right]$$

$$= \frac{x^2}{4} + \frac{1}{2}\left[\frac{x\sin 2x}{2} + \frac{\cos 2x}{4}\right] + C$$

$$= \frac{x^2}{4} + \frac{x\sin 2x}{4} + \frac{\cos 2x}{8} + C$$

P1:

$$\int x sin^2 x\, dx =$$

Solution:

$$\int x\sin^2x\,dx = \int \frac{x(1-\cos2x)}{2}dx$$

$$= \frac{1}{2}\int x\,dx - \frac{1}{2}\int x\cos2x\ dx$$

$$= \frac{x^2}{4} - \frac{1}{2}\int x\cos2x\ dx \text{ .... } (1)$$

Now, to find $\int x\cos2x\ dx$

Put $u = x$ and $dv = \cos2x dx$

$\Longrightarrow du = dx$ and $v = \int \cos2x dx = \frac{\sin2x}{2}$

By integration by parts, we have

$$\int u dv = uv - \int v\,dv$$

$$\int x\cos2x\ dx = \frac{x\sin2x}{2} - \int \frac{\sin2x}{2}dx$$

$$= \frac{x\sin2x}{2} + \frac{\cos2x}{4} + C'$$

where C' is an arbitrary constant

From (1), we have

$$\therefore \int x\sin^2x\,dx = \frac{x^2}{4} - \frac{1}{2}\int x\cos2x\ dx$$

$$= \frac{x^2}{4} - \frac{1}{2}\left[\frac{x\sin2x}{2} + \frac{\cos2x}{4}\right] + C'$$

$$= \frac{x^2}{4} - \frac{x\sin2x}{4} + \frac{\cos2x}{8} + C'$$

IP2:

$$\int e^{-x}\cos 2x \ dx =$$

Solution:

Step1:

Given $\int e^{-x}\cos 2x \ dx$

Put $u = e^{-x}$ and $dv = \cos 2x \ dx$

$\Rightarrow du = -e^{-x}dx$ and $v = \frac{\sin 2x}{2}$

Step2:

$$\int e^{-x}\cos 2x \ dx = \frac{e^{-x}\sin 2x}{2} - \int \frac{\sin 2x}{2}(-e^{-x}dx)$$

$$= \frac{e^{-x}\sin 2x}{2} + \frac{1}{2}\int e^{-x} \sin 2x \, dx \ldots\ldots (1)$$

Step3:

Now, to find $\int e^{-x} \sin 2x \, dx$

Put $u = e^{-x}$ and $dv = \sin 2x dx$

$\Rightarrow du = -e^{-x}d$ and $v = \int \sin 2x dx = -\frac{\cos 2x}{2}$

By integration by parts, we have

$$\int u dv = uv - \int v \, dv$$

$$\int e^{-x} \sin 2x \, dx = -\frac{e^{-x}\cos 2x}{2} + \int \frac{\cos 2x}{2}(-e^{-x}dx)$$

$$= -\frac{e^{-x}\cos 2x}{2} - \frac{1}{2}\int e^{-x}\cos 2x \, dx$$

From (1), we have

$$\therefore \int e^{-x}\cos 2x \ dx$$

$$= \frac{e^{-x}\sin 2x}{2} + \frac{1}{2}\left[-\frac{e^{-x}\cos 2x}{2} - \frac{1}{2}\int e^{-x}\cos 2x \, dx\right]$$

$$= \frac{e^{-x}\sin 2x}{2} - \frac{e^{-x}\cos 2x}{4} - \frac{1}{4}\int e^{-x} \cos 2x \, dx$$

$$\left(1 + \frac{1}{4}\right)\int e^{-x}\sin 2x \ dx = \frac{e^{-x}}{4}(2\sin 2x - \cos 2x) + C'$$

$$\int e^{-x}\sin 2x \ dx = \frac{e^{-x}}{5}(2\sin 2x - \cos 2x) + C$$

Where C is an arbitrary constant

Solution:

Given $\int e^{-x} sin2x\ dx$

Put $u = e^{-x}$ and $dv = sin2x\,dx$

$\Longrightarrow du = -e^{-x}dx$ and $v = \int sin2x\,dx = -\frac{cos2x}{2}$

By integration by parts, we have

$$\int udv = uv - \int v\ dv$$

$$\int e^{-x} sin2x\ dx = -\frac{e^{-x}cos2x}{2} - \int \left(-\frac{cos2x}{2}\right)(-e^{-x}dx)$$

$$= -\frac{e^{-x}cos2x}{2} - \frac{1}{2}\int e^{-x}\,cos2x\,dx \text{ ………..} (1)$$

Now, to find $\int e^{-x}\,cos2x\,dx$

Put $u = e^{-x}$ and $dv = cos2x\,dx$

$\Longrightarrow du = -e^{-x}dx$ and $v = \int cos2x\,dx = \frac{sin2x}{2}$

By integration by parts, we have

$$\int udv = uv - \int v\ dv$$

$$\int e^{-x}\,cos2x\,dx = \frac{e^{-x}sin2x}{2} - \int \frac{sin2x}{2}(-e^{-x}dx)$$

$$= \frac{e^{-x}sin2x}{2} + \frac{1}{2}\int e^{-x}\,sin2x\,dx$$

From (1), we have

$$\therefore \int e^{-x} sin2x\ dx = -\frac{e^{-x}cos2x}{2} - \frac{1}{2}\int e^{-x}\,cos2x\,dx$$

$$= -\frac{e^{-x}cos2x}{2} - \frac{e^{-x}sin2x}{4} - \frac{1}{4}\int e^{-x}\,sin2x\,dx$$

$$\left(1 + \frac{1}{4}\right)\int e^{-x} sin2x\ dx = -\frac{e^{-x}}{4}(2cos2x + sin2x) + C'$$

$$\int e^{-x} sin2x\ dx = -\frac{e^{-x}}{5}(2cos2x + sin2x) + C$$

where C is an arbitrary constant

IP3:

$$\int \boldsymbol{x} \ \mathbf{log}(\boldsymbol{x}+\mathbf{1}) \ \boldsymbol{dx} =$$

Solution:

Step1:

Given $\int x \ \log(x+1) \ dx$

$u = x$ and $dv = \log(x+1)\,dx$

$\Longrightarrow du = dx$ and

$v = \int \log(x+1)dx = (x+1)[\log(x+1) - 1]$

Step2:

By integration by parts, we have

$$\int u dv = uv - \int v\, dv$$

$$\begin{aligned}\int x \ \log(x+1) \ dx &= x(x+1)[\log(x+1)-1] - \int (x+1)[\log(x+1)-1] \ dx \\ &= x(x+1)[\log(x+1)-1] - \int (x+1)\log(x+1) \ dx \\ &\qquad + \int (x+1)\,dx \\ &= x(x+1)[\log(x+1)-1] - \int x \log(x+1) \ dx \\ &\qquad - \int \log(x+1)\,dx + \frac{x^2}{2} + x\end{aligned}$$

$$\begin{aligned}2\int x \ \log(x+1) \ dx &= x(x+1)[\log(x+1)-1] - (x+1)[\log(x+1)-1] \\ &\qquad + \frac{x^2}{2} + x + C \\ &= \frac{1}{2}\left[(x-1)(x+1)[\log(x+1)-1] + \frac{x^2}{2} + x + C\right] \\ &= \frac{1}{2}\left[(x^2-1)[\log(x+1)-1] + \frac{x^2}{2} + x + C\right] \\ &= \frac{1}{2}\left[(x^2-1)\log(x+1) - x^2 + 1 + \frac{x^2}{2} + x + C\right]\end{aligned}$$

$$\int x \ \log(x+1) \ dx = \frac{1}{2}\left[(x^2-1)\log(x+1) - \frac{x^2}{2} + x + 1\right] + K$$

where K is an arbitrary constant

P3:

$$\int \sqrt{x}\, \log x \; dx =$$

Solution:

Given $\int \sqrt{x}\ \log x\ \ dx$

Put $u = \sqrt{x}$ and $dv = \log x\, dx$

$\Rightarrow du = \frac{1}{2\sqrt{x}}\ dx$ and $v = x(\log x - 1)$

By integration by parts, we have

$$\int u dv = uv - \int v\ dv$$

$$\int \sqrt{x}\ \log x\ \ dx = x\sqrt{x}(\log x - 1) - \int x(\log x - 1)\frac{1}{2\sqrt{x}} dx$$

$$= x^{\frac{3}{2}}(\log x - 1) - \frac{1}{2}\int \sqrt{x} \log x\ \ dx + \frac{1}{2}\int \sqrt{x}\ \ dx$$

$$\therefore \left(\frac{1}{2} + 1\right)\int \sqrt{x}\ \log x\ \ dx = x^{\frac{3}{2}}(\log x - 1) + \frac{1}{2}.\frac{2}{3}\ x^{\frac{3}{2}} + C'$$

$$\int \sqrt{x}\ \log x\ \ dx = \frac{2}{3}\left[x^{\frac{3}{2}}\ \log x - x^{\frac{3}{2}} + \frac{1}{3}\ x^{\frac{3}{2}}\right] + C$$

where C is an arbitrary constant

$$= \frac{2}{3}x^{\frac{3}{2}}(\log x - 1) - \frac{2}{3}\ x^{\frac{3}{2}}\left(1 - \frac{1}{3}\right) + C$$

$$\int \sqrt{x}\ \log x\ \ dx = \frac{2}{3}x^{\frac{3}{2}}(\log x - 1) - \frac{4}{9}\ x^{\frac{3}{2}} + C$$

IP4:

Evaluate $\int (x^3 - 5x^2 - 9x + 12)e^{-x}\ dx$ by tabular integration

Solution:

Step1:

Given $\int (x^3 - 5x^2 - 9x + 12)e^{-x}\ dx$

Take $f(x) = x^3 - 5x^2 - 9x + 12$ and $g(x) = e^{-x}$

Step2:

Table

$f(x)$ and its derivatives		$g(x)$ and its integrals
$x^3 - 5x^2 - 9x + 12$		e^{-x}
	$(+)$	
$3x^2 - 10x - 9$		$-e^{-x}$
	$(-)$	
$6x - 10$		e^{-x}
	$(+)$	
6		$-e^{-x}$
	$(-)$	
0		e^{-x}

Step3:

By tabular integration,

$$\int (x^3 - 5x^2 - 9x + 12)e^{-x}\ dx$$

$$= (x^3 - 5x^2 - 9x + 12)(-e^{-x}) - (3x^2 - 10x - 9)e^{-x}$$

$$+(6x - 10)(-e^{-x}) - 6e^{-x} + C$$

where C is an arbitrary constant

$$= -e^{-x}(x^3 - 5x^2 - 9x + 12 + 3x^2 - 10x - 9 + 6x - 10 + 6)$$

$$= -e^{-x}(x^3 - 2x^2 - 13x - 1) + C$$

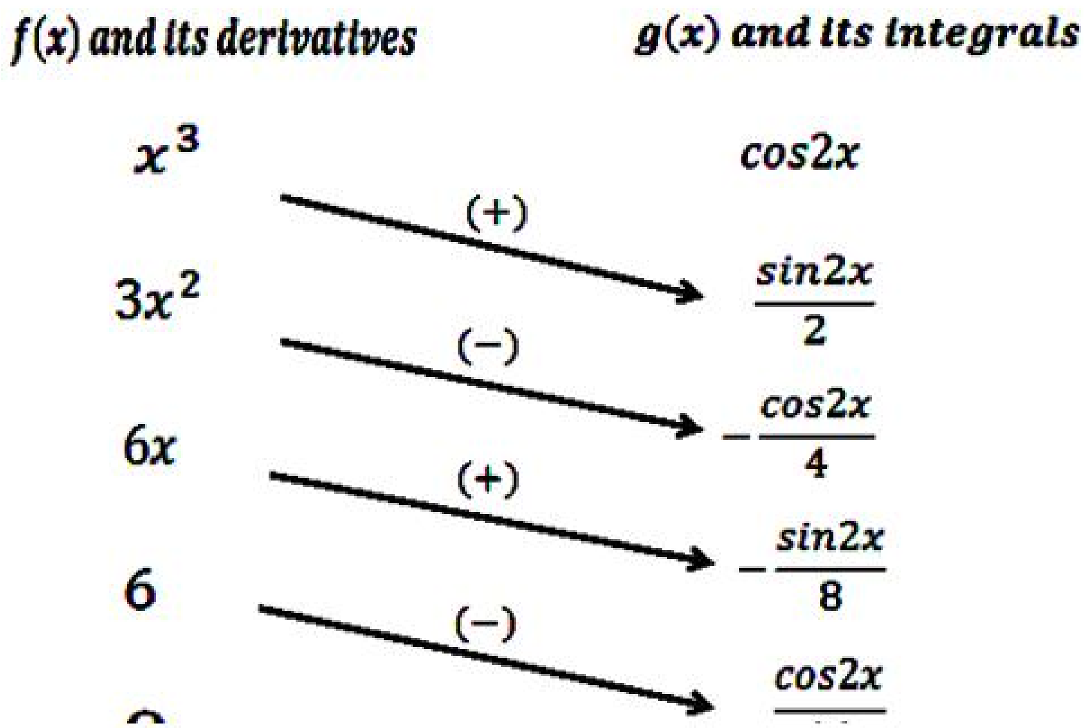
P4:

Evaluate $\int x^3 \cos 2x \, dx$ by tabular integration

Solution:

Take $f(x) = x^3$ and $g(x) = cos2x$

Table



By tabular integration,

$$\therefore \int x^3 cos2x \ dx$$

$$= \frac{x^3 \, sin2x}{2} - 3x^2\left(-\frac{cos2x}{4}\right) + 6x\left(-\frac{sin2x}{8}\right) - 6\left(\frac{cos2x}{16}\right) + C$$

where C is an arbitrary constant

$$= \frac{x^3 \, sin2x}{2} + \frac{3x^2 \, cos2x}{4} - \frac{3x \, sin2x}{4} - \frac{3cos2x}{8} + C$$

HWA-22(11.1&11.2)

Answer all the questions and submit

11.1

1. Find the integrals of the following functions:

   a. $f(x) = \frac{1+\sin^2 x}{1+cos2x}$

   b. $f(x) = \frac{1}{4x^2+4x+5}$

   c. $f(x) = \frac{cosx}{\sqrt{4-\sin^2 x}}$

   d. $f(x) = \frac{x^2+3x+4}{x+2}$

   e. $f(x) = \frac{x^2}{\sqrt{1+x^2}\left(1+\sqrt{1+x^2}\right)}$

11.2

2. Find the integrals of the following functions:

   a. $f(x) = \frac{e^x(1-sinx)}{1-cosx}$

   b. $f(x) = (x^3 - 2x^2 + 3)e^{3x}$

   c. $f(x) = x^4 sin2x$

   d. $f(x) = x^3\, ln(1+x)$

1. Evaluate

a. $\int x \sin^{-1} x \, dx$

b. $\int x \, cosx \, dx$

c. $\int x \, sinx \, dx$

d. $\int x \tan^{-1} x \, dx$

e. $\int \log(1+x^2) \, dx$

f. $\int x^2 \log x \, dx$

g. $\int \sin^{-1} x \, dx$

h. $\int \cos^{-1} x \, dx$

i. $\int \tan^{-1} x \, dx$

j. $\int \log\left(x+\sqrt{x^2+a^2}\right) \, dx$

2. Evaluate

a. $\int e^{2x} sin4x$

b. $\int e^{-5x} cos4x$

c. $\int e^{-2x}\, sin3x\, dx$

d. $\int t^3 \log(1+t)\, dt$

e. $\int x^2 \log x\, dt$

3. Evaluate by tabular integration

a. $\int x^7 e^{3x}\, dx$

b. $\int x^{10} e^{6x}\, dx$

c. $\int (x^5 + 2x^3 + 5x^2 + x + 1)^2 e^x\, dx$

d. $\int x^3 (\log x)^2\, dx$

e. $\int x^5 sin2x\, dx$

f. $\int x^4 cos2x\, dx$

12.3 Partial Fractions

Learning objectives:

- To express the rational fractions into simpler fractions by the method of partial fractions.
- To evaluate the integrals by the method of partial fractions in which the integrand consists of
  - Distinct linear factors in the denominator.
  - A repeated linear factor in the denominator.
  - An irreducible quadratic factor in the denominator.

  AND
- To practice the related problems.

A theorem from advanced algebra says that every rational function, no matter how complicated, can be rewritten as a sum of simpler fractions that we can integrate with techniques we already know. For instance,

$$\frac{5x-3}{x^2-2x-3}=\frac{2}{x+1}+\frac{3}{x-3} \quad \text{............ (1)}$$

So, we can integrate the rational function on the left by integrating the fractions on the right instead.

The method for rewriting rational functions in this way is called the ***method of partial fractions.*** In this particular case, it consists of finding constants A and B such that

$$\frac{5x-3}{x^2-2x-3}=\frac{A}{x+1}+\frac{B}{x-3} \quad \text{............. (2)}$$

We call the fractions $A/(x+1)$ and $B/(x-3)$ as ***partial fractions*** because their denominators are only part of the original denominator x^2-2x-3. We call A and B ***undetermined coefficients*** until proper values for them have been found.

To find A and B, we first clear the equation (2) of fractions, obtaining

$$5x-3=A(x-3)+B(x+1)=(A+B)x-3A+B$$

This will be an identity in x if and only if the coefficients of like powers of x on the two sides are equal:

$$A+B=5\ ,\ -3A+B=-3$$

Solving these equations simultaneously gives $A=2$ and $B=3$.

Example 1- ***Two distinct linear factors in the denominator***

Find

$$\int\frac{5x-3}{(x+1)(x-3)}dx$$

Solution

$$\int\frac{5x-3}{(x+1)(x-3)}dx=\int\frac{2}{x+1}dx+\int\frac{3}{x-3}dx$$
$$=2\ln|x+1|+3\ln|x-3|+C$$

where C is an arbitrary constant

Example 2- ***A repeated linear factor in the denominator***

Express

$$\frac{6x+7}{(x+2)^2}$$

as a sum of partial fractions and evaluate $\int\frac{6x+7}{(x+2)^2}\,dx$

Solution

Since the denominator has a repeated linear factor, $(x+2)^2$, we must express the fraction in the form

$$\frac{6x+7}{(x+2)^2}=\frac{A}{x+2}+\frac{B}{(x+2)^2} \quad \text{.......... (3)}$$

Clearing equation (3) of fractions gives

$$6x+7=A(x+2)+B=Ax+(2A+B)$$

Matching the coefficients of like terms gives $A=6$ and

$$\therefore 7=2A+B=12+B\Rightarrow B=-5$$

Hence $$\frac{6x+7}{(x+2)^2}=\frac{6}{x+2}-\frac{5}{(x+2)^2}$$

$$Now,\ \int\frac{6x+7}{(x+2)^2}\,dx=\int\left(\frac{6}{x+2}-\frac{5}{(x+2)^2}\right)dx$$
$$=6\int\frac{dx}{x+2}-5\int\frac{dx}{(x+2)^2}$$
$$=6\log|x+2|+\frac{5}{x+2}+C$$

where C is an arbitrary constant

--

Example 3 ***An improper fraction***

Express

$$\frac{2x^3-4x^2-x-3}{x^2-2x-3}$$

as a sum of partial fractions and evaluate $\int\frac{2x^3-4x^2-x-3}{x^2-2x-3}\,dx$

Solution

First we divide the denominator into the numerator to get a polynomial plus a proper fraction. Then we write the proper fraction as a sum of partial fractions. Long division gives

$$\frac{2x^3-4x^2-x-3}{x^2-2x-3}=2x+\frac{5x-3}{x^2-2x-3}$$
$$=2x+\frac{2}{x+1}+\frac{3}{x-3}$$

$$\int\frac{2x^3-4x^2-x-3}{x^2-2x-3}\,dx=\int\left(2x+\frac{2}{x+1}+\frac{3}{x-3}\right)dx$$
$$=2\int x\,dx+2\int\frac{dx}{x+1}+3\int\frac{dx}{x-3}$$
$$=x^2+2\log|x+1|+3\log|x-3|+C$$

where C is an arbitrary constant

Example 4- ***An irreducible quadratic factor in the denominator***

Express

$$\frac{-2x+4}{\left(x^2+1\right)(x-1)^2}$$

as a sum of partial fractions

Solution

The denominator has an irreducible quadratic factor as well as a repeated linear factor, so we write

$$\frac{-2x+4}{\left(x^2+1\right)(x-1)^2}=\frac{Ax+B}{x^2+1}+\frac{C}{x-1}+\frac{D}{(x-1)^2} \quad \text{......... (4)}$$

For quadratic factors, we use first degree numerators, not constant numerators. Clearing the equation of fractions gives

$$-2x+4=(Ax+B)(x-1)^2+C(x-1)\left(x^2+1\right)+D\left(x^2+1\right)$$
$$=(A+C)x^3+(-2A+B-C+D)x^2$$
$$+(A-2B+C)x+(B-C+D)$$

Equating coefficients of like terms gives

Coefficients of x^3: $\quad 0=A+C$
Coefficients of x^2: $\quad 0=-2A+B-C+D$
Coefficients of x^1: $\quad -2=A-2B+C$
Coefficients of x^0: $\quad 4=B-C+D$

We solve these equations simultaneously to find the values of A, B, C, and D.

$$A=2 \qquad C=-2 \qquad B=1 \qquad D=1$$

We substitute these values into equation (4), obtaining

$$\frac{-2x+4}{\left(x^2+1\right)(x-1)^2}=\frac{2x+1}{x^2+1}-\frac{2}{x-1}+\frac{1}{(x-1)^2}$$

Evaluate $\int\frac{-2x+4}{\left(x^2+1\right)(x-1)^2}dx$

Solution We expand the integrand by partial fractions, as in example 4, and integrate the terms of the expansion:

$$\int\frac{-2x+4}{\left(x^2+1\right)(x-1)^2}dx=\int\left(\frac{2x+1}{x^2+1}-\frac{2}{x-1}+\frac{1}{(x-1)^2}\right)dx$$
$$=\int\left(\frac{2x}{x^2+1}+\frac{1}{x^2+1}-\frac{2}{x-1}+\frac{1}{(x-1)^2}\right)dx$$
$$=\ln\left(x^2+1\right)+\tan^{-1}x-2\ln|x-1|-\frac{1}{x-1}+C$$

where C is an arbitrary constant

IP1.

$$\int \frac{3x-2}{(x-1)(x^2-x-6)}\,dx =$$

Solution:

Step1:

Let $\frac{3x-2}{(x-1)(x+2)(x-3)} = \frac{A}{(x-1)} + \frac{B}{(x+2)} + \frac{C}{(x-3)}$

$\Rightarrow (3x-2)$

$= A(x+2)(x-3) + B(x-1)(x-3) + C(x-1)(x+2)$

.... (1)

Set $x = 1$ in (1), we have

$3-2 = A(1+2)(1-3) \Rightarrow A = -\frac{1}{6}$

Set $x = -2$ in (1) , we have

$-6-2 = B(-2-1)(-2-3) \Rightarrow B = -\frac{8}{15}$

Set $x = 3$ in (1), we have

$9-2 = C(3-1)(3+2) \Rightarrow C = \frac{7}{10}$

Step2:

$$\frac{3x-2}{(x-1)(x+2)(x-3)} = -\frac{1}{6(x-1)} - \frac{8}{15(x+2)} + \frac{7}{10(x-3)}$$

Step3:

$$\int \frac{3x-2}{(x-1)(x+2)(x-3)}\,dx$$

$$= -\frac{1}{6}\int \frac{1}{(x-1)}\,dx - \frac{8}{15}\int \frac{1}{(x+2)}\,dx + \frac{7}{10}\int \frac{1}{(x-3)}\,dx$$

$$= -\frac{1}{6}\log|x-1| - \frac{8}{15}\log|x+2| + \frac{7}{10}\log|x-3| + C$$

where C is an arbitrary constant

P1.

$$\int \frac{x^2 + 4x - 7}{(x^2 - x - 12)(x + 1)}\, dx =$$

Solution:

$$\frac{x^2+4x-7}{(x^2-x-12)(x+1)} = \frac{x^2+4x-7}{(x-4)(x+1)(x+3)}$$

Let $\frac{x^2+4x-7}{(x-4)(x+1)(x+3)} = \frac{A}{(x-4)} + \frac{B}{(x+1)} + \frac{C}{(x+3)}$

$\Rightarrow x^2 + 4x - 7$

$= A(x+1)(x+3) + B(x-4)(x+3) + C(x-4)(x+1)$

.... (1)

Set $x = 4$ in (1), we get

$16 + 16 - 7 = A(4+1)(4+3) \Rightarrow A = \frac{5}{7}$

Set $x = -1$ in (1), we get

$1 - 4 - 7 = B(-1-4)(-1+3) \Rightarrow B = 1$

Set $x = -3$ in (1), we get

$9 - 12 - 7 = C(-3-4)(-3+1) \Rightarrow C = -\frac{5}{7}$

$$\frac{x^2+4x-7}{(x-4)(x+1)(x+3)} = \frac{5}{7(x-4)} + \frac{1}{(x+1)} - \frac{5}{7(x+3)}$$

$$\int \frac{x^2+4x-7}{(x-4)(x+1)(x+3)} dx$$

$$= \frac{5}{7}\int \frac{dx}{(x-4)} + \int \frac{dx}{(x+1)} - \frac{5}{7}\int \frac{dx}{(x+3)}$$

$$= \frac{5}{7}\log|x-4| + \log|x+1| - \frac{5}{7}\log|x+3| + C$$

where C is an arbitrary constant

IP2.

$$\int \frac{x^2}{(x+1)(x+2)^2}\,dx =$$

Solution:

Step1:

Let $\frac{x^2}{(x+1)(x+2)^2} = \frac{A}{(x+1)} + \frac{B}{(x+2)} + \frac{C}{(x+2)^2}$

$$\Rightarrow x^2 = A(x+2)^2 + B(x+1)(x+2) + C(x+1)$$

……………. (1)

Step2:

Put $x = -1$ in (1), we get

$(-1)^2 = A(-1+2)^2 \Rightarrow A = 1$

Put $x = -2$ in (1), we get

$(-2)^2 = C(-2+1) \Rightarrow C = -4$

Now, comparing the coefficients of x^2 on both sides, we get

$$1 = A + B \Rightarrow B = 0$$

Step3:

Hence $\frac{x^2}{(x+1)(x+2)^2} = \frac{1}{(x+1)} - \frac{4}{(x+2)^2}$

Step4:

$$\int \frac{x^2}{(x+1)(x+2)^2}\,dx = \int \frac{dx}{x+1} - \int \frac{4}{(x+2)^2}dx$$

$$= \log|x+1| + \frac{4}{x+2} + C$$

where C is an arbitrary constant

P2.

$$\int \frac{3x+1}{(x+3)(x-1)^2}\,dx =$$

Solution:

Let $\frac{3x+1}{(x+3)(x-1)^2} = \frac{A}{(x+3)} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2}$

$$\Rightarrow 3x + 1 = A(x-1)^2 + B(x+3)(x-1) + C(x+3)$$

.... (1)

Set $x = -3$ in (1), we get

$$-9 + 1 = A(-3-1)^2 \Rightarrow A = -\frac{1}{2}$$

Set $x = 1$ in (1), we get

$$3 + 1 = C(1+3) \Rightarrow C = 1$$

Now, comparing the coefficients of x^2 on both sides of (1),

we get

$$A + B = 0 \Rightarrow B = \frac{1}{2}$$

$$\therefore \frac{3x+1}{(x+3)(x-1)^2} = -\frac{1}{2(x+3)} + \frac{1}{2(x-1)} + \frac{1}{(x-1)^2}$$

$$\int \frac{3x+1}{(x+3)(x-1)^2}dx = -\frac{1}{2}\int \frac{dx}{(x+3)} + \frac{1}{2}\int \frac{dx}{(x-1)} + \int \frac{dx}{(x-1)^2}$$

$$= -\frac{1}{2}\log|x+3| + \frac{1}{2}\log|x-1| - \frac{1}{(x-1)} + C$$

$$= \frac{1}{2}\log\left|\frac{x-1}{x+3}\right| - \frac{1}{(x-1)} + C$$

where C is an arbitrary constant

IP3.

$$\int \frac{16x^3}{4x^2 - 4x + 1}\, dx =$$

Solution:

Step1:

$$\frac{16x^3}{4x^2-4x+1} = (4x+4) + \frac{12x-4}{4x^2-4x+1}$$

$(\because \textit{it is an improper fraction})$

Step2:

$$\int \frac{16x^3}{4x^2-4x+1}\, dx = 4\int x\, dx + 4\int dx + \int \frac{12x-4}{4x^2-4x+1}\, dx$$

$$= 2x^2 + 4x + \int \frac{12x-4}{(2x-1)^2}\, dx \text{ ...... } (1)$$

Step3:

Now, we have to evaluate $\int \frac{12x-4}{(2x-1)^2}\, dx$

Let $\frac{12x-4}{(2x-1)^2} = \frac{A}{(2x-1)} + \frac{B}{(2x-1)^2}$

$12x - 4 = A(2x-1) + B$ .... (2)

Put $x = \frac{1}{2}$ in (2), we get

$B = 6 - 4 = 2$

Comparing the coefficients of x on both sides of (2), we get

$12 = 2A \Longrightarrow A = 6$

$$\frac{12x-4}{(2x-1)^2} = \frac{6}{(2x-1)} + \frac{2}{(2x-1)^2}$$

$$\int \frac{12x-4}{(2x-1)^2} dx = 6\int \frac{dx}{(2x-1)} + 2\int \frac{dx}{(2x-1)^2}$$

$$= 6\ \frac{\log|2x-1|}{2} - 2\ \frac{\left(\frac{1}{2x-1}\right)}{2}$$

$$\int \frac{12x-4}{(2x-1)^2} dx = 3\log|2x-1| - \frac{1}{2x-1}$$

Step4:

$$(1) \Longrightarrow \int \frac{16x^3}{4x^2-4x+1}\, dx$$

$$= 2x^2 + 4x + 3\log|2x-1| - \frac{1}{2x-1} + C$$

where C is an arbitrary constant

P3.

$$\int \frac{x^3 - 2x + 3}{x^2 + x - 2}\,dx =$$

Solution:

$$\frac{x^3-2x+3}{x^2+x-2} = x - 1 + \frac{x+1}{x^2+x-2} \ (\because it\ is\ an\ improper\ fraction)$$

$$\int \frac{x^3-2x+3}{x^2+x-2}\,dx = \int x\,dx - \int dx + \int \frac{x+1}{x^2+x-2}\,dx$$

$$= \frac{x^2}{2} - x + \int \frac{x+1}{x^2+x-2}\,dx \quad \text{..........} (1)$$

Now, $\int \frac{x+1}{x^2+x-2}\,dx = \int \frac{x+1}{(x-1)(x+2)}\,dx$

Writing as the sum of partial fractions,

$$\frac{x+1}{(x-1)(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x+2)} \quad \text{..............} (2)$$

$$\Rightarrow (x+1) = A(x+2) + B(x-1)$$

Put $x = 1$ in (2), we get

$$2 = A(1+2) \Rightarrow A = \frac{2}{3}$$

Put $x = -2$ in (2), we get

$$-2 + 1 = B(-2-1) \Rightarrow B = \frac{1}{3}$$

$$(2) \Rightarrow \frac{x+1}{(x-1)(x+2)} = \frac{2}{3(x-1)} + \frac{1}{3(x+2)}$$

$$\therefore \int \frac{x+1}{(x-1)(x+2)}\,dx = \frac{2}{3}\int \frac{dx}{(x-1)} + \frac{1}{3}\int \frac{dx}{(x+2)}$$

$$= \frac{2}{3}\log|x-1| + \frac{1}{3}\log|x+2| + C$$

From (1),

$$\int \frac{x^3-2x+3}{x^2+x-2}\,dx$$

$$= \frac{x^2}{2} - x + \frac{2}{3}\log|x-1| + \frac{1}{3}\log|x+2| + C$$

where C is an arbitrary constant

$$= \frac{x^2}{2} - x + \log\left|(x-1)^{\frac{2}{3}}\right| + \log\left|(x+2)^{\frac{1}{3}}\right| + C$$

$$= \frac{x^2}{2} - x + \log\left|(x-1)^{\frac{2}{3}}.(x+2)^{\frac{1}{3}}\right| + C$$

IP4.

$$\int \frac{\mathbf{2x^2+3x-2}}{\mathbf{(x-3)^2(x^2+16)}}\,\mathbf{dx} =$$

Solution:

Step1:

Here (x^2+16) is an irreducible quadratic factor

Let $\frac{2x^2+3x-2}{(x-3)^2(x^2+16)} = \frac{A}{(x-3)} + \frac{B}{(x-3)^2} + \frac{Cx+D}{(x^2+16)}$

$\Longrightarrow 2x^2+3x-2$

$= A(x^2+16)(x-3) + B(x^2+16) + (Cx+D)(x-3)^2$ ………….. (1)

Set $x = 3$ in (1), we get

$2(3)^2 + 3(3) - 2 = B(9+16) \Longrightarrow B = 1$

Simplifying (1), we get

$$2x^2+3x-2 = x^3(A+C) + x^2(-3A+B-6C+D)$$
$$+x(16A+9C-6D) + (-48A+16B+9D)$$

Equating the coefficients of like terms on both sides, we get

x^3 Coefficients: $A + C = 0$

x^2 Coefficients: $-3A + B - 6C + D = 2$

x Coefficients: $16A + 9C - 6D = 3$

x^0 Coefficients: $-48A + 16B + 9D = -2$

Solving the above equations, we get

$A = \frac{9}{25},\ B = 1,\ C = -\frac{9}{25},\ D = -\frac{2}{25}$

$$\frac{2x^2+3x-2}{(x-3)^2(x^2+16)} = \frac{\frac{9}{25}}{(x-3)} + \frac{1}{(x-3)^2} - \frac{\frac{9}{25}x+\frac{2}{25}}{(x^2+16)}$$

Step2:

$\int \frac{2x^2+3x-2}{(x-3)^2(x^2+16)}dx$

$$= \frac{9}{25}\int \frac{dx}{(x-3)} + \int \frac{dx}{(x-3)^2} - \int \frac{\frac{9}{25}x+\frac{2}{25}}{(x^2+16)}dx$$

$$= \frac{9}{25}\log|x-3| - \frac{1}{(x-3)} - \frac{9}{50}\int \frac{2x}{(x^2+16)}dx - \frac{2}{25}\int \frac{dx}{4^2+x^2}$$

$$= \frac{9}{25}\log|x-3| - \frac{1}{(x-3)} - \frac{9}{50}\log|x^2+16| - \frac{1}{50}\tan^{-1}\frac{x}{4} + C$$

$$= \frac{9}{25}\log\left|\frac{x-3}{\sqrt{x^2+16}}\right| - \frac{1}{(x-3)} - \frac{1}{50}\tan^{-1}\frac{x}{4} + C$$

where C is an arbitrary constant

P4.

$$\int \frac{dx}{(x+1)^2(x^2+1)} =$$

Solution:

Here x^2+1 is an irreducible quadratic factor,

Let $\frac{1}{(x+1)^2(x^2+1)} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{Cx+D}{(x^2+1)}$

$$\Rightarrow 1 = A(x+1)(x^2+1) + B(x^2+1) + (Cx+D)(x+1)^2$$

…………… (1)

Set $x=-1$ in (1), we get

$1 = B(1+1) \Rightarrow B = \frac{1}{2}$

Simplifying (1), we get

$$1 = A(x^3+x^2+x+1) + B(x^2+1) + (Cx+D)(x^2+2x+1)$$

$$1 = A(x^3+x^2+x+1) + B(x^2+1) + C(x^3+2x^2+x)$$
$$+D(x^2+2x+1)$$

$$1 = (A+C)x^3 + (A+B+2C+D)x^2$$
$$+(A+C+2D)x + (A+B+D)$$

Equating the coefficients of like terms, we get

x^3 Coefficients: $A+C=0$

x^2 Coefficients: $A+B+2C+D=0$

x Coefficients: $A+C+2D=0$

x^0 Coefficients: $B+D+A=1$

Solving above equations, we get

$A=\frac{1}{2},\;\; C=-\frac{1}{2},\;\; D=0$

Hence $\frac{1}{(x+1)^2(x^2+1)} = \frac{1}{2(x+1)} + \frac{1}{2(x+1)^2} - \frac{x}{2(x^2+1)}$

$$\int \frac{dx}{(x+1)^2(x^2+1)} = \frac{1}{2}\int \frac{dx}{(x+1)} + \frac{1}{2}\int \frac{dx}{(x+1)^2} - \frac{1}{2}\int \frac{x\,dx}{(x^2+1)}$$

$$= \frac{1}{2}\log|x+1| - \frac{1}{2}\frac{1}{(x+1)} - \frac{1}{2}.\frac{1}{2}\int \frac{2x\,dx}{(x^2+1)}$$

$$= \frac{1}{2}\log|x+1| - \frac{1}{2(x+1)} - \frac{1}{4}\log|x^2+1| + C$$

where C is an arbitrary constant

$$= \frac{1}{2}\log\left|\frac{x+1}{\sqrt{x^2+1}}\right| - \frac{1}{2(x+1)} + C$$

1. Evaluate

a. $\int \frac{2x+1}{x^2-7x+12}\,dx$

b. $\int \frac{x+4}{x^2+5x-6}\,dx$

c. $\int \frac{y}{y^2-2y-3}\,dy$

d. $\int \frac{2x+1}{(x^2-4x+3)(x^2-x+30)}\,dx$

e. $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$

f. $\int_1^2 \frac{4y^2-7y-12}{y^3-y^2-6y}\,dy$

2. Evaluate

A. $\int_0^1 \frac{x^3}{x^2+2x+1}\,dx$

B. $\int_{-1}^0 \frac{x^3}{x^2-2x+1}\,dx$

C. $\int \frac{x^2}{(x-1)(x^2+2x+1)}\,dx$

D. $\int \frac{2x^2+x+1}{(x+3)(x-2)^2}\,dx$

E. $\int \frac{2x^2-5x+1}{x^2(x^2-1)}\,dx$

F. $\int \frac{7x-4}{(x+2)(x-1)^2}\,dx$

3. Evaluate

a. $\int \frac{2x^3-2x^2+1}{x^2-x}\,dx$

b. $\int \frac{9x^3-3x+1}{x^3-x^2}\,dx$

c. $\int \frac{x^3+x}{x-1}\,dx$

d. $\int \frac{x^3-2x^2+4x+1}{x^3-x^2-x+1}\,dx$

e. $\int \frac{4x^2-3x+2}{4x^2-4x+3}\,dx$

4. Evaluate

a. $\int \frac{y^4+y^2-1}{y^3+y}\, dy$

b. $\int \frac{2y^4}{y^3-y^2+y-1}\, dy$

c. $\int \frac{2x^2-x+4}{x^3+4x}\, dx$

d. $\int \frac{2x+1}{(x+1)^3(x^2+4)^2}\, dx$

e. $\int \frac{x^2-x+6}{x^3+3x}\, dx$

f. $\int \frac{x^3+x^2+2x+1}{(x^2+1)(x^2+2)}\, dx$

12.4

General Description of the Method of Partial Fractions

Learning objectives:

- To illustrate the method of writing a proper fraction as the sum of partial fractions.
- To illustrate Heaviside cover up method for writing a proper fraction with distinct linear factors into sum of partial fractions.
- To illustrate the methods of determining the undetermined coefficients by differentiation and assigning selected numerical values.
  AND
- To practice the related problems.

Success in writing a rational function $f(x)/g(x)$ as a sum of partial fractions depends on two things:

1. *The degree of $f(x)$ must be less than the degree of* $g(x)$. If it is not, we divide and work with the remainder term.

2. *We must know the factors of* $g(x)$. In theory, any polynomial with real coefficients can be written as a product of real linear factors and real quadratic factors. In practice, the factors may be hard to find.

A theorem from advanced algebra says that when these two conditions are met, we may write $f(x)/g(x)$ as the sum of partial fractions by taking these steps.

The Method of Partial Fractions ($f(x)/g(x)$ Proper)

Step 1: Let $x-r$ be a linear factor of $g(x)$. Suppose $(x-r)^m$ is the highest power of $x-r$ that divides $g(x)$.

Then assign the sum of m partial fractions to this factor, as follows:

$$\frac{A_1}{x-r}+\frac{A_2}{(x-r)^2}+\cdots+\frac{A_m}{(x-r)^m}$$

We do this for each distinct linear factor of $g(x)$.

Step 2: Let x^2+px+q be an irreducible quadratic factor of $g(x)$. Suppose $\left(x^2+px+q\right)^n$ is the highest power of this factor that divides $g(x)$. Then to this factor assign the sum of the n partial fractions:

$$\frac{B_1x+C_1}{x^2+px+q}+\frac{B_2x+C_2}{\left(x^2+px+q\right)^2}+\cdots+\frac{B_nx+C_n}{\left(x^2+px+q\right)^n}$$

We do this for each distinct irreducible quadratic factor

Step 3: Set the original fraction $f(x)/g(x)$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x.

Step 4: Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

The Heaviside "Cover-up" Method for Linear factors

When the degree of the polynomial $f(x)$ is less than the degree of $g(x)$, and

$$g(x)=(x-r_1)(x-r_2)\cdots(x-r_n)$$

is a product of n *distinct linear factors*, each raised to the first power, there is a quick way to expand $f(x)/g(x)$ by partial fractions.

Example1

Find A, B, and C in the partial-fraction expansion

$$\frac{x^2+1}{(x-1)(x-2)(x-3)}=\frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{x-3} \quad \text{.............. (1)}$$

Solution:

If we multiply both sides of equation (1) by $(x-1)$ to get

$$\frac{x^2+1}{(x-2)(x-3)}=A+\frac{B(x-1)}{x-2}+\frac{C(x-1)}{x-3}$$

and set $x=1$, the resulting equation gives the value of A:

$$\frac{1^2+1}{(1-2)(1-3)}=A+0+0\Rightarrow A=1$$

Thus, the value of A is the number we would have obtained if we had covered the factor $(x-1)$ in the denominator of the original fraction

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} \quad \text{.................. (2)}$$

and evaluated the rest at $x=1$.

$$A=\frac{(1)^2+1}{\boxed{(x-1)}(1-2)(1-3)}=\frac{2}{(-1)(-2)}=1$$

↑
Cover

Similarly, we find the value of B in equation (1) by covering the factor $(x-2)$ in (2) and evaluating the rest at $x=2$:

$$B=\frac{2^2+1}{(2-1)(2-3)}=\frac{5}{(1)(-1)}=-5$$

Finally, C is found by covering the $(x-3)$ in (2) and evaluating the rest at $x=3$:

$$C=\frac{3^2+1}{(3-1)(3-2)}=\frac{10}{(2)(1)}=5$$

Heaviside Method:

The following are the steps in the cover-up method:

Step 1: *Write the quotient with $g(x)$ factored:*

$$\frac{f(x)}{g(x)}=\frac{f(x)}{(x-r_1)(x-r_2)\cdots(x-r_n)} \quad \text{............. (3)}$$

Step 2: *Cover the factors $(x-r_i)$ of $g(x)$ in (3) one at a time,* each time replacing all the uncovered x's by the number r_i. This gives a number A_i for each root r_i:

$$A_1=\frac{f(r_1)}{(r_1-r_2)\cdots(r_1-r_n)}$$

$$A_2=\frac{f(r_2)}{(r_2-r_1)(r_2-r_3)\cdots(r_2-r_n)}$$

$$\vdots$$

$$A_n=\frac{f(r_n)}{(r_n-r_1)(r_n-r_2)\cdots(r_n-r_{n-1})}$$

Step 3: *Write the partial-fraction expansion of $f(x)/g(x)$ as*

$$\frac{f(x)}{g(x)}=\frac{A_1}{(x-r_1)}+\frac{A_2}{(x-r_2)}+\cdots+\frac{A_n}{(x-r_n)}$$

Example2

Evaluate using Heaviside Cover-up method,

$$\int\frac{x+4}{x^3+3x^2-10x}dx$$

Solution:

The degree of $\boldsymbol{f(x)=x+4}$ is less than the degree of $\boldsymbol{g(x)=x^3+3x^2-10x}$, and with $\boldsymbol{g(x)}$ factored ,

$$\frac{x+4}{x^3+3x^2-10x}=\frac{x+4}{x(x-2)(x+5)}$$

The roots of $g(x)$ are $r_1=0, r_2=2$ *and* $r_3=-5$. We find

$$A_1=\frac{0+4}{(0-2)(0+5)}=\frac{4}{(-2)(5)}=-\frac{2}{5}$$

$$A_2=\frac{2+4}{(2)(2+5)}=\frac{6}{(2)(7)}=\frac{3}{7}$$

$$A_3=\frac{-5+4}{(-5)(5-2)}=\frac{-1}{(-5)(-7)}=-\frac{1}{35}$$

Therefore,

$$\frac{x+4}{x(x-2)(x+5)}=-\frac{2}{5x}+\frac{3}{7(x-2)}-\frac{1}{35(x+5)}$$

and

$$\int\frac{x+4}{x^3+3x^2-10x}dx=-\frac{2}{5}ln|x|+\frac{3}{7}ln|x-2|-\frac{1}{35}ln|x+5|+C$$

Other Ways to Determine the Constants

Another way to determine the constants that appear in partial fractions is to differentiate, as in the next example. Still another is to assign selected numerical values to x.

Example 3 ***Using Differentiation***

Find $A, B, and\ C$ in the equation

$$\frac{x-1}{(x+1)^3}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{(x+1)^3}$$

Solution:

We first clear of fractions:

$$x-1=A(x+1)^2+B(x+1)+C$$

Substituting $x=-1$ shows $C=-2$. We then differentiate both sides with respect to x, obtaining

$$1=2A(x+1)+B$$

Substituting $x=-1$ shows $B=1$. We differentiate again to get $0=2A$, which shows $A=0$. Hence

$$\frac{x-1}{(x+1)^3}=\frac{1}{(x+1)^2}-\frac{2}{(x+1)^3}$$

In some problems, assigning small values to x such as $x=0,\pm1,\pm2$, to get equations in $A, B,$ and C provides a fast alternative to other methods.

Example 4 **Assigning numerical values to x**

Find $A, B, and\ C$ in

$$\frac{x^2+1}{(x-1)(x-2)(x-3)}=\frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{x-3}$$

Solution

Clear of fractions to get

$$x^2+1=A(x-2)(x-3)+B(x-1)(x-3)$$
$$+C(x-1))x-2)$$

Then let $x=1,2{,}3$ successively to find A, B, and C:

$x=1$: $\quad (1)^2+1=A(-1)(-2)+B(0)+C(0)$
$\quad \Rightarrow 2=2A\Rightarrow A=1$

$x=2$: $\quad (2)^2+1=A(0)+B(1)(-1)+C(0)$
$\quad \Rightarrow 5=-B\Rightarrow B=-5$

$x=3$: $\quad (3)^2+1=A(0)+B(0)+C(2)(1)$
$\quad \Rightarrow 10=2C\Rightarrow C=5$

Thus

$$\frac{x^2+1}{(x-1)(x-2)(x-3)}=\frac{1}{x-1}-\frac{5}{x-2}+\frac{5}{x-3}$$

IP1.

$$\int \frac{4x-5}{(x^2-4x+3)(x^2-x-2)}\,dx =$$

Solution:

Step1:

The degree of $f(x) = 4x - 5$ is less than the degree of $g(x) = (x^2 - 4x + 3)(x^2 - x - 2)$ and by factoring $g(x)$,

we have

$$\frac{4x-5}{(x^2-4x+3)(x^2-x-2)} = \frac{4x-5}{(x+1)(x-1)(x-2)(x-3)} \;\text{.........}\; (1)$$

Step2:

By Heaviside cover-up method, we have

$$\frac{4x-5}{(x+1)(x-1)(x-2)(x-3)} = \frac{A_1}{(x+1)} + \frac{A_2}{(x-1)} + \frac{A_3}{(x-2)} + \frac{A_4}{(x-3)}$$

The roots of $g(x)$ are $r_1 = -1,\ r_2 = 1,\ r_3 = 2,\ r_4 = 3.$

$$A_1 = \frac{-4-5}{(-1-1)(-1-2)(-1-3)} \Rightarrow A_1 = \frac{3}{8}$$

$$A_2 = \frac{4-5}{(1+1)(1-2)(1-3)} \Rightarrow A_2 = -\frac{1}{4}$$

$$A_3 = \frac{8-5}{(2+1)(2-1)(2-3)} \Rightarrow A_3 = -1$$

$$A_4 = \frac{12-5}{(3+1)(3-1)(3-2)} \Rightarrow A_4 = \frac{7}{8}$$

$$\therefore \frac{4x-5}{(x+1)(x-1)(x-2)(x-3)} = \frac{3}{8(x+1)} - \frac{1}{4(x-1)} - \frac{1}{(x-2)} + \frac{7}{8(x-3)}$$

Step3:

$$\int \frac{4x-5}{(x+1)(x-1)(x-2)(x-3)}\,dx$$

$$= \frac{3}{8}\int \frac{dx}{(x+1)} - \frac{1}{4}\int \frac{dx}{(x-1)} - \int \frac{dx}{(x-2)} + \frac{7}{8}\int \frac{dx}{(x-3)}$$

$$= \frac{3}{8}\log|x+1| - \frac{1}{4}\log|x-1| - \log|x-2|$$

$$+\frac{7}{8}\log|x-3| + C$$

$$= \frac{1}{8}(3\log|x+1| + 7\log|x-3|)$$

$$-\left(\frac{1}{4}\log|x-1| + \log|x-2|\right) + C$$

where C is an arbitrary constant

$$= \frac{1}{8}\log|(x+1)^3.(x-3)^7| - \log\left|(x-1)^{\frac{1}{4}}.(x-2)\right| + C$$

P1.

$$\int \frac{x^2 - 5x + 2}{(x^2 - 1)(x^2 - 4)}\, dx =$$

Solution:

The degree of $f(x) = x^2 - 5x + 2$ is less than the degree of $g(x) = (x^2 - 1)(x^2 - 4)$ and by factoring $g(x)$, we have

$$\frac{x^2-5x+2}{(x^2-1)(x^2-4)} = \frac{x^2-5x+2}{(x-1)(x+1)(x-2)(x+2)}$$

$$\frac{x^2-5x+2}{(x-1)(x+1)(x-2)(x+2)} = \frac{A_1}{(x-1)} + \frac{A_2}{(x+1)} + \frac{A_3}{(x-2)} + \frac{A_4}{(x+2)}$$

The roots of $g(x)$ are $r_1 = 1,\ r_2 = -1,\ r_3 = 2,\ r_4 = -2$

By Heaviside cover-up method,

$$A_1 = \frac{1-5+2}{(1+1)(1-2)(1+2)} \Rightarrow A_1 = \frac{1}{3}$$

$$A_2 = \frac{1+5+2}{(-1-1)(-1-2)(-1+2)} \Rightarrow A_2 = \frac{4}{3}$$

$$A_3 = \frac{4-10+2}{(2-1)(2+1)(2+2)} = -\frac{1}{3}$$

$$A_4 = \frac{4+10+2}{(-2-1)(-2+1)(-2-2)} = -\frac{4}{3}$$

$$\therefore\ \frac{x^2-5x+2}{(x-1)(x+1)(x-2)(x+2)} = \frac{1}{3(x-1)} + \frac{4}{3(x+1)} - \frac{1}{3(x-2)} - \frac{4}{3(x+2)}$$

$$\int \frac{x^2-5x+2}{(x^2-1)(x^2-4)}dx$$

$$= \frac{1}{3}\int \frac{dx}{(x-1)} + \frac{4}{3}\int \frac{dx}{(x+1)} - \frac{1}{3}\int \frac{dx}{(x-2)} - \frac{4}{3}\int \frac{dx}{(x+2)}$$

$$= \frac{1}{3}\log|x-1| + \frac{4}{3}\log|x+1| - \frac{1}{3}\log|x-2| - \frac{4}{3}\log|x+2| + C$$

where C is an arbitrary constant

$$= \frac{1}{3}\log\left|\frac{x-1}{x-2}\right| + \frac{4}{3}\log\left|\frac{x+1}{x+2}\right| + C$$

IP2.

$$\int \frac{2\theta^3 + 5\theta^2 + 8\theta + 4}{(\theta^2 + 2\theta + 2)^2}\, d\theta =$$

Solution:

Here $(\theta^2 + 2\theta + 2)$ is an irreducible factor

Step1:

$$\frac{2\theta^3+5\theta^2+8\theta+4}{(\theta^2+2\theta+2)^2} = \frac{B_1\theta+C_1}{\theta^2+2\theta+2} + \frac{B_2\theta+C_2}{(\theta^2+2\theta+2)^2}$$

$$2\theta^3 + 5\theta^2 + 8\theta + 4$$

$$= (B_1\theta + C_1)(\theta^2 + 2\theta + 2) + (B_2\theta + C_2)$$

$$= B_1\theta^3 + (2B_1 + C_1)\theta^2 + (2B_1 + B_2 + 2C_1)\theta + (2C_1 + C_2)$$

Step2:

Equating the coefficients of like terms on both sides, we get

θ^3 Coefficients: $B_1 = 2$

θ^2 Coefficients: $2B_1 + C_1 = 5$

θ^1 Coefficients: $2B_1 + B_2 + 2C_1 = 8$

θ^0 Coefficients: $2C_1 + C_2 = 4$

Solving the above equations, we get

$B_1 = 2,\ B_2 = 2,\ C_1 = 1,\ C_2 = 2$

$$\therefore\ \frac{2\theta^3+5\theta^2+8\theta+4}{(\theta^2+2\theta+2)^2} = \frac{2\theta+1}{\theta^2+2\theta+2} + \frac{2\theta+2}{(\theta^2+2\theta+2)^2}$$

Step3:

$$\int \frac{2\theta^3+5\theta^2+8\theta+4}{(\theta^2+2\theta+2)^2} d\theta$$

$$= \int \frac{2\theta+1}{\theta^2+2\theta+2} d\theta + \int \frac{2\theta+2}{(\theta^2+2\theta+2)^2} d\theta$$

$$= \int \frac{(2\theta+2)-1}{\theta^2+2\theta+2} d\theta + \int \frac{2\theta+2}{(\theta^2+2\theta+2)^2} d\theta$$

$$= \int \frac{2\theta+2}{\theta^2+2\theta+2} d\theta - \int \frac{d\theta}{\theta^2+2\theta+2} + \int \frac{2\theta+2}{(\theta^2+2\theta+2)^2} d\theta$$

$$= \int \frac{2\theta+2}{\theta^2+2\theta+2} d\theta - \int \frac{d\theta}{(\theta+1)^2+(1)^2} + \int \frac{2\theta+2}{(\theta^2+2\theta+2)^2} d\theta$$

$$= \int \frac{2\theta+2}{\theta^2+2\theta+2} d\theta - \tan^{-1}(\theta+1) + \int \frac{2\theta+2}{(\theta^2+2\theta+2)^2} d\theta$$

Put $\boldsymbol{\theta^2 + 2\theta + 2 = t \Rightarrow (2\theta + 2)d\theta = dt}$

$$= \int \frac{1}{t} dt - \tan^{-1}(\theta + 1) + \int \frac{1}{t^2} dt$$

$$= \log|t| - \tan^{-1}(\theta + 1) - \frac{1}{t} + C$$

where C is an arbitrary constant

$$= \log|\theta^2 + 2\theta + 2| - \tan^{-1}(\theta + 1) - \frac{1}{\theta^2+2\theta+2} + C$$

P2.

$$\int \frac{\theta^4 - 4\theta^3 + 2\theta^2 - 3\theta + 1}{(\theta^2 + 1)^3} d\theta =$$

Solution:

Here $\left(\theta^2+1\right)$ is an irreducible factor

$$\frac{\theta^4-4\theta^3+2\theta^2-3\theta+1}{(\theta^2+1)^3}=\frac{B_1\theta+C_1}{(\theta^2+1)}+\frac{B_2\theta+C_2}{(\theta^2+1)^2}+\frac{B_3\theta+C_3}{(\theta^2+1)^3}$$

$$\theta^4-4\theta^3+2\theta^2-3\theta+1$$

$$=(B_1\theta+C_1)(\theta^2+1)^2+(B_2\theta+C_2)(\theta^2+1)+(B_3\theta+C_3)$$

$$=B_1\theta^5+C_1\theta^4+(2B_1+B_2)\theta^3+(2C_1+C_2)\theta^2$$

$$+(B_1+B_2+B_3)\theta+(C_1+C_2+C_3)$$

Equating the coefficients of like terms on both sides, we get

θ^5 Coefficients: $B_1=0$

θ^4 Coefficients: $C_1=1$

θ^3 Coefficients: $2B_1+B_2=-4$

θ^2 Coefficients: $2C_1+C_2=2$

θ^1 Coefficients: $B_1+B_2+B_3=-3$

θ^0 Coefficients: $C_1+C_2+C_3=1$

Solving the above equations, we get

$B_1=0, B_2=-4, B_3=1, C_1=1, C_2=0, C_3=0$

$$\frac{\theta^4-4\theta^3+2\theta^2-3\theta+1}{(\theta^2+1)^3}=\frac{1}{(\theta^2+1)}-\frac{4\theta}{(\theta^2+1)^2}+\frac{\theta}{(\theta^2+1)^3}$$

$$\int\frac{\theta^4-4\theta^3+2\theta^2-3\theta+1}{(\theta^2+1)^3}\,d\theta$$

$$=\int\frac{1}{(\theta^2+1)}\,d\theta-4\int\frac{\theta}{(\theta^2+1)^2}\,d\theta+\int\frac{\theta}{(\theta^2+1)^3}\,d\theta$$

$$=\tan^{-1}\theta-2\int\frac{2\theta}{(\theta^2+1)^2}\,d\theta+\frac{1}{2}\int\frac{2\theta}{(\theta^2+1)^3}\,d\theta$$

Put $\boldsymbol{\theta^2+1=t\Rightarrow 2\theta\,d\theta=dt}$

$$=\tan^{-1}\theta-2\int\frac{dt}{t^2}+\frac{1}{2}\int\frac{dt}{t^3}$$

$$=\tan^{-1}\theta+\frac{2}{t}+\frac{1}{2}\left(-\frac{1}{2t^2}\right)+C$$

where C is an arbitrary constant

$$=\tan^{-1}\theta+\frac{2}{(\theta^2+1)}-\frac{1}{4(\theta^2+1)^2}+C$$

IP3.

$$\int \frac{2x+1}{x(x^2+4)^2}\,dx =$$

Solution:

$$\frac{2x+1}{x(x^2+4)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+4} + \frac{Dx+E}{(x^2+4)^2}$$

$$2x+1 = A(x^2+4)^2 + (Bx+C)x(x^2+4) + (Dx+E)x \text{ .... } (1)$$

Set $x = 0$ in (1), we get

$$0+1 = A(0+4)^2 \Longrightarrow A = \frac{1}{16}$$

Now,

$$2x+1 = (A+B)x^4 + Cx^3 + (8A+4B+D)x^2 + (4C+E)x$$

Equating the coefficients of like terms on both sides, we get

x^4 Coefficients: $A+B=0$

x^3 Coefficients: $C=0$

x^2 Coefficients: $8A+4B+D=0$

x^1 Coefficients: $4C+E=2$

Solving the above equations, we get

$$A = \frac{1}{16},\ B = -\frac{1}{16},\ C = 0,\ D = -\frac{1}{4}, E = 2$$

$$\therefore\ \frac{2x+1}{x(x^2+4)^2} = \frac{1}{16x} - \frac{\frac{x}{16}}{x^2+4} + \frac{-\frac{x}{4}+2}{(x^2+4)^2}$$

$$\int \frac{2x+1}{x(x^2+4)^2}\,dx$$

$$= \frac{1}{16}\int\frac{dx}{x} - \frac{1}{16}\int\frac{x}{x^2+4}\,dx - \frac{1}{4}\int\frac{x}{(x^2+4)^2}\,dx + 2\int\frac{dx}{(x^2+4)^2}$$

$$= \frac{1}{16}\int\frac{dx}{x} - \frac{1}{32}\int\frac{2x}{x^2+4}\,dx - \frac{1}{8}\int\frac{2x}{(x^2+4)^2}\,dx + 2\int\frac{dx}{(x^2+4)^2}$$

$$= \frac{1}{16}\log|x| - \frac{1}{32}\log|x^2+4| + \frac{1}{8(x^2+4)} + 2\int\frac{dx}{(x^2+4)^2} + C_1$$

.... (2)

Now, we have to evaluate $\int \frac{dx}{(x^2+4)^2}$

Put $x = 2tan\theta \Longrightarrow dx = 2sec^2\theta\, d\theta,\ \theta = \tan^{-1}\frac{x}{2}$

$$\int\frac{dx}{(x^2+4)^2} = \int\frac{2sec^2\theta}{(4tan^2\theta+4)^2}d\theta = \int\frac{2sec^2\theta}{4^2(tan^2\theta+1)^2}d\theta$$

$$= \frac{2}{16}\int\frac{sec^2\theta}{sec^4\theta}d\theta = \frac{1}{8}\int\frac{d\theta}{sec^2\theta} = \frac{1}{8}\int cos^2\theta\ d\theta$$

$$= \frac{1}{16}\int(1+cos2\theta)\,d\theta = \frac{1}{16}\left[\theta + \frac{sin2\theta}{2}\right] + C_2$$

$$= \frac{1}{16}\left[\theta + \frac{tan\theta}{1+tan^2\theta}\right] + C_2 = \frac{1}{16}\left[\tan^{-1}\frac{x}{2} + \frac{2x}{x^2+4}\right] + C_2$$

.... (3)

Substituting (3) in (2), we have

$$\int\frac{2x+1}{x(x^2+4)^2}\,dx$$

$$= \frac{1}{16}\log|x| - \frac{1}{32}\log|x^2+4| + \frac{1}{8(x^2+4)}$$

$$+\frac{2}{16}\left[\tan^{-1}\frac{x}{2} + \frac{2x}{x^2+4}\right] + C$$

$$= \frac{1}{16}\log|x| - \frac{1}{32}\log|x^2+4| + \frac{1}{8(x^2+4)}$$

$$+\frac{1}{8}\tan^{-1}\frac{x}{2} + \frac{1}{4}\left[\frac{x}{x^2+4}\right] + C$$

where $C_1 + C_2 = C$ is an arbitrary constant

P3.

$$\int \frac{dx}{x(x^2+1)^2} =$$

Solution:

Here (x^2+1) is an irreducible factor

Let $\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$

$1 = A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x$

$= A(x^4+2x^2+1) + B(x^4+x^2) + C(x^3+x) + (Dx^2+Ex)$

$= (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A$

Equating the coefficients of like terms on both sides, we get

x^4 Coefficients: $A+B=0$

x^3 Coefficients: $C=0$

x^2 Coefficients: $2A+B+D=0$

x^1 Coefficients: $C+E=0$

x^0 Coefficients: $A=1$

Solving above equations, we get

$A=1,\ \ B=-1,\ \ C=0,\ \ D=-1,\ \ E=0$

$\therefore\ \frac{1}{x(x^2+1)^2} = \frac{1}{x} - \frac{x}{x^2+1} - \frac{x}{(x^2+1)^2}$

$$\int \frac{dx}{x(x^2+1)^2} = \int \frac{dx}{x} - \int \frac{x}{x^2+1}\,dx - \int \frac{x}{(x^2+1)^2}\,dx$$

$$= \log|x| - \int \frac{x}{x^2+1}\,dx - \int \frac{x}{(x^2+1)^2}\,dx$$

$$= \log|x| - \frac{1}{2}\int \frac{2x}{x^2+1}\,dx - \frac{1}{2}\int \frac{2x}{(x^2+1)^2}\,dx$$

Put $\boldsymbol{x^2+1=t \Rightarrow 2x\,dx=dt}$

$$= \log|x| - \frac{1}{2}\int \frac{du}{u} - \frac{1}{2}\int \frac{du}{u^2}$$

$$= \log|x| - \frac{1}{2}\log|u| + \frac{1}{2u} + C$$

where C is an arbitrary constant

$$= \log|x| - \frac{1}{2}\log|x^2+1| + \frac{1}{2(x^2+1)} + C$$

$$= \log\left|\frac{x}{\sqrt{x^2+1}}\right| + \frac{1}{2(x^2+1)} + C$$

IP4.

$$\int \frac{2 - x^2}{(x + 5)^3}\, dx =$$

Solution:

Step1:

Let $\frac{2-x^2}{(x+5)^3} = \frac{A}{(x+5)} + \frac{B}{(x+5)^2} + \frac{C}{(x+5)^3}$

$2 - x^2 = A(x+5)^2 + B(x+5) + C$ .... (1)

Set $x = -5$ in (1), we have $2 - (-5)^2 = C \Rightarrow C = -23$

Differentiating (1) w.r.t x on both sides, we get

$-2x = 2A(x+5) + B$ .... (2)

Set $x = -5$ in (2), we have $-2(-5) = B \Rightarrow B = 10$

Differentiating (2) w.r.t x on both sides, we get

$-2 = 2A \Rightarrow A = -1$

$$\therefore \frac{2-x^2}{(x+5)^3} = -\frac{1}{(x+5)} + \frac{10}{(x+5)^2} - \frac{23}{(x+5)^3}$$

Step2:

$$\int \frac{2-x^2}{(x+5)^3}\, dx = -\int \frac{dx}{(x+5)} + 10\int \frac{dx}{(x+5)^2} - 23\int \frac{dx}{(x+5)^3}$$

$$= -\log|x+5| - \frac{10}{x+5} + \frac{23}{2(x+5)^2} + C$$

$$= \log\left|\frac{1}{(x+5)}\right| - \frac{10}{x+5} + \frac{23}{2(x+5)^2} + C$$

where C is an arbitrary constant

P4.

$$\int \frac{3x^3 + 9x^2 + x}{(x-1)^4}\, dx =$$

Solution:

$$\frac{3x^3+9x^2+x}{(x-1)^4} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{(x-1)^4}$$

$$3x^3 + 9x^2 + x = A(x-1)^3 + B(x-1)^2 + C(x-1) + D$$

$$\ldots\ldots\ldots\ldots\ (1)$$

Set $x = 1$ in (1), we get

$3 + 9 + 1 = D \Rightarrow D = 13$

Differentiating (1) w.r.t x on both sides, we get

$9x^2 + 18x + 1 = 3A(x-1)^2 + 2B(x-1) + C$ .... (2)

Set $x = 1$ in (2), we get

$9 + 18 + 1 = C \Rightarrow C = 28$

Differentiating (2) w.r.t x on both sides, we get

$18x + 1 = 6A(x-1) + 2B$ .... (3)

Set $x = 1$ in (3), we get

$18 + 18 = 2B \Rightarrow B = 18$

Differentiating (3) w.r.t x on both sides, we get

$18 = 6A \Rightarrow A = 3$

$$\therefore \frac{3x^3+9x^2+x}{(x-1)^4} = \frac{3}{(x-1)} + \frac{18}{(x-1)^2} + \frac{28}{(x-1)^3} + \frac{13}{(x-1)^4}$$

$$\int \frac{3x^3+9x^2+x}{(x-1)^4}dx$$

$$= 3\int \frac{dx}{(x-1)} + 18\int \frac{dx}{(x-1)^2} + 28\int \frac{dx}{(x-1)^3} + 13\int \frac{dx}{(x-1)^4}$$

$$= 3\log|x-1| - \frac{18}{x-1} - \frac{14}{(x-1)^2} - \frac{13}{3(x-1)^3} + C$$

where C is an arbitrary constant

1. Evaluate

a. $\int \frac{2x+3}{(x^2-16)(x+9)}dx$

b. $\int \frac{3x^3-4x^2+5}{(x^2-5x+6)(x^2-9x+20)}dx$

c. $\int \frac{5x^2+9}{(x+6)(x^2-9x+18)(x^2-4x-21)}dx$

d. $\int \frac{2x^2}{(x^2-1)(x^2-4)(x^2-9)}dx$

e. $\int \frac{3t^2+t+4}{t^3+t}dt$

f. $\int \frac{dx}{x(x+1)(x+2)}$

2. Evaluate

a. $\int \frac{3x-5}{x(x^2+2x+4)} dx$

b. $\int \frac{y^2+2y+1}{(y^2+1)^2} dy$

c. $\int \frac{8x^2+8x+2}{(4x^2+1)^2} dx$

d. $\int \frac{2s+2}{(s^2+1)(s-1)^3} ds$

e. $\int \frac{dx}{(x+1)(x^2+1)^2}$

f. $\int \frac{2x+3}{(x+3)(x^2+4)^3} dx$

g. $\int \frac{x}{(x^2+4)(x^2+9)} dx$

3. Evaluate

a. $\int \frac{x^3+5x+4}{(x-1)^5}\,dx$

b. $\int \frac{3x+4}{(x+4)^5}\,dx$

c. $\int \frac{4x^2+5x+1}{(x-6)^6}\,dx$

d. $\int \frac{3x^2-3x+4}{(x-2)^3}\,dx$

e. $\int \frac{x^3-3x}{(x-4)^4}\,dx$

f. $\int \frac{3x^2+5}{(x-7)^5}\,dx$

12.5

Trigonometric substitutions

Learning objectives:

* To evaluate integrals involving

$$\sqrt{a^2+x^2}, \sqrt{a^2-x^2}, \text{ and } \sqrt{x^2-a^2}$$

by substitutions

$$x = a\tan\theta, x = a\sin\theta, \text{ and } x = a\sec\theta$$

respectively.

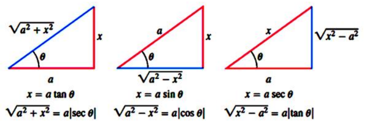
AND

* To practice related problems.

Trigonometric substitutions can be effective in transforming integrals involving $\sqrt{a^2+x^2}, \sqrt{a^2-x^2}$, and $\sqrt{x^2-a^2}$ into integrals which can be evaluated directly.

Three Basic Substitutions

The most common substitutions are $x = a\tan\theta$, $x = a\sin\theta$, and $x = a\sec\theta$; They come from the following reference right triangles.



With $x = a\tan\theta$,

$$a^2+x^2 = a^2+a^2\tan^2\theta = a^2(1+\tan^2\theta) = a^2\sec^2\theta$$

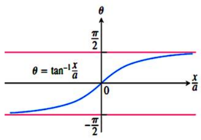
With $x = a\sin\theta$,

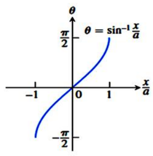
$$a^2-x^2 = a^2-a^2\sin^2\theta = a^2(1-\sin^2\theta) = a^2\cos^2\theta$$

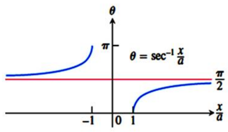
With $x = a\sec\theta$,

$$x^2-a^2 = a^2\sec^2\theta - a^2 = a^2(\sec^2\theta-1) = a^2\tan^2\theta$$

The following are the graphs of arc tangent, arc sine and arc secant of $\frac{x}{a}$.







To change back to the original variable, we set

$$\theta = \tan^{-1}\left(\frac{x}{a}\right) \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$\theta = \sin^{-1}\left(\frac{x}{a}\right) \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$\theta = \sec^{-1}\left(\frac{x}{a}\right) \qquad 0 \le \theta < \frac{\pi}{2} \text{ if } \frac{x}{a} \ge 1$$

$$\frac{\pi}{2} < \theta \le \pi \text{ if } \frac{x}{a} \le -1$$

With the substitution $x = a\sec\theta$, we will restrict its use to integrals in which $\frac{x}{a} \ge 1$; this will place θ in $[0, \frac{\pi}{2})$ and make $\tan\theta \ge 0$. We will then have $\sqrt{x^2-a^2} = \sqrt{a^2\tan^2\theta} = |a\tan\theta| = a\tan\theta$, free of absolute values, provided $a > 0$.

Example 1

Evaluate $\int \frac{dx}{\sqrt{4+x^2}}$

Solution

We set $x = 2\tan\theta \Rightarrow dx = 2\sec^2\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

Now, $4+x^2 = 4+4\tan^2\theta = 4\left(1+\tan^2\theta\right) = 4\sec^2\theta$. Then

$$\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{2\sec^2\theta\, d\theta}{\sqrt{4\sec^2\theta}} = \int \frac{\sec^2\theta\, d\theta}{|\sec\theta|} \qquad \left(\because \sqrt{\sec^2\theta} = |\sec\theta|\right)$$

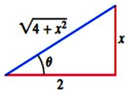
$$= \int \sec\theta\, d\theta \qquad \left(\because \sec\theta > 0 \quad \text{for} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$$

$$= \ln|\sec\theta + \tan\theta| + C'$$

$$= \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C'$$

$$= \ln\left|\sqrt{4+x^2} + x\right| + C$$

To express $\ln|\sec\theta + \tan\theta|$ in terms of x, we draw a reference triangle for the original substitution $x = 2\tan\theta$ and read the ratios from the triangle.



Substitution of $x = a\sin\theta$

The following example illustrate the substitution $x = a\sin\theta$

Example 2

Evaluate $\int \frac{x^2 dx}{\sqrt{9-x^2}}$

Solution

To replace $9 - x^2$ by a single squared term, we set

$$x = 3\sin\theta \Rightarrow dx = 3\cos\theta\, d\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$Now,\ 9 - x^2 = 9\left(1-\sin^2\theta\right) = 9\cos^2\theta$$

Then

$$\int \frac{x^2 dx}{\sqrt{9-x^2}} = \int \frac{9\sin^2\theta \cdot 3\cos\theta\, d\theta}{|3\cos\theta|} \qquad \left(\because\ \sqrt{\cos^2\theta} = |\cos\theta|\right)$$

$$= 9\int \sin^2\theta\, d\theta \qquad \left(\because\ \cos\theta > 0 \quad \text{for} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$$

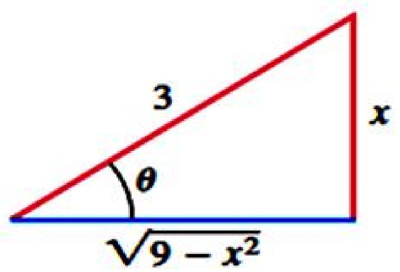
$$= 9\int \frac{1-\cos 2\theta}{2} d\theta$$

$$= \frac{9}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) + C$$

$$= \frac{9}{2}\left(\theta - \sin\theta\cos\theta\right) + C$$

$$= \frac{9}{2}\left(\sin^{-1}\frac{x}{3} - \frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}\right) + C$$

$$= \frac{9}{2}\sin^{-1}\frac{x}{3} - \frac{x}{2}\sqrt{9-x^2} + C$$



Substitution of $x = a\,sec\theta$

The following example illustrate the substitution $x = a\,sec\theta$

Example 3

Evaluate $\int \frac{dx}{\sqrt{25x^2-4}}$, $x > \frac{2}{5}$

Solution

We first rewrite the radical as

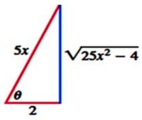
$$\sqrt{25x^2-4} = \sqrt{25\left(x^2-\frac{4}{25}\right)} = 5\sqrt{x^2-\left(\frac{2}{5}\right)^2}$$

to put the radicand in the form of $\sqrt{x^2-a^2}$. We then substitute

$$x = \frac{2}{5}\sec\theta \Rightarrow dx = \frac{2}{5}\sec\theta\tan\theta\,d\theta, \quad 0<\theta<\frac{\pi}{2}$$

$$x^2-\left(\frac{2}{5}\right)^2 = \frac{4}{25}\sec^2\theta - \frac{4}{25} = \frac{4}{25}\left(\sec^2\theta-1\right) = \frac{4}{25}\tan^2\theta$$

$$\sqrt{x^2-\left(\frac{2}{5}\right)^2} = \frac{2}{5}\left|\tan\theta\right| = \frac{2}{5}\tan\theta \qquad \left(\because\ \tan\theta>0 \quad \text{for} \quad 0<\theta<\frac{\pi}{2}\right)$$



With these substitutions, we have

$$\int \frac{dx}{\sqrt{25x^2-4}} = \int \frac{dx}{5\sqrt{x^2-(4/25)}}$$
$$= \int \frac{(2/5)\sec\theta\tan\theta\,d\theta}{5\cdot(2/5)\tan\theta}$$
$$= \frac{1}{5}\int \sec\theta\,d\theta = \frac{1}{5}\ln\left|\sec\theta+\tan\theta\right| + C$$
$$= \frac{1}{5}\ln\left|\frac{5x}{2}+\frac{\sqrt{25x^2-4}}{2}\right| + C$$

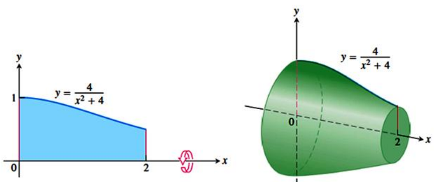
A trigonometric substitution can sometimes help us to evaluate an integral containing an ***integral power of a quadratic binomial***, as in the next example.

Example 4

Find the volume of the solid generated by revolving about the x-axis the region bounded by the curve $y = 4/(x^2+4)$, the x-axis, and the lines $x = 0$ and $x = 2$.

Solution

We sketch the region, figure below, and use the disk method.

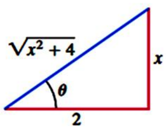


$$V = \int_0^2 \pi\left[R(x)\right]^2 dx = 16\pi\int_0^2 \frac{dx}{\left(x^2+4\right)^2} \quad , \quad R(x) = \frac{4}{x^2+4}$$

To evaluate the integral, we set

$$x = 2\tan\theta \Rightarrow dx = 2\sec^2\theta\,d\theta \quad and \quad \theta = \tan^{-1}\frac{x}{2}$$

$$Now,\ x^2+4 = 4\tan^2\theta+4 = 4\left(\tan^2\theta+1\right) = 4\sec^2\theta$$



With these substitutions

$$V = 16\pi\int_0^2 \frac{dx}{\left(x^2+4\right)^2}$$
$$= 16\pi\int_0^{\pi/4} \frac{2\sec^2\theta\,d\theta}{\left(4\sec^2\theta\right)^2}$$
$$= 16\pi\int_0^{\pi/4} \frac{2\sec^2\theta\,d\theta}{16\sec^4\theta}$$
$$= \pi\int_0^{\pi/4} 2\cos^2\theta\,d\theta$$
$$= \pi\int_0^{\pi/4} (1+\cos 2\theta)\,d\theta$$
$$= \pi\left[\theta+\frac{\sin 2\theta}{2}\right]_0^{\pi/4} = \pi\left[\frac{\pi}{4}+\frac{1}{2}\right] \approx 4.04$$

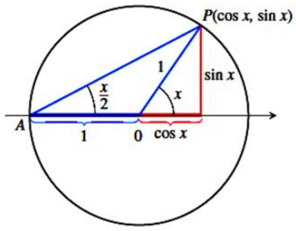
The Substitution $z = tan\frac{x}{2}$:

The substitution

$$z = tan\frac{x}{2}$$

reduces the problem of integrating a rational expression in $sinx$ and $cosx$ to a problem of integrating a rational function of z. This in turn can be integrated by partial fractions.

From the accompanying figure,



We can read the relation $tan\frac{x}{2} = \frac{sinx}{1+cosx}$

To see the effect of the substitution, we calculate

$$cosx = 2cos^2\left(\frac{x}{2}\right) - 1 = \frac{2}{sec^2\left(\frac{x}{2}\right)} - 1$$

$$= \frac{2}{1+tan^2\left(\frac{x}{2}\right)} - 1 = \frac{2}{1+z^2} - 1 = \frac{1-z^2}{1+z^2}$$

$$\therefore\ \boldsymbol{cosx = \frac{1-z^2}{1+z^2}}$$

and

$$sinx = 2sin\frac{x}{2}cos\frac{x}{2} = \frac{2sin\frac{x}{2}}{cos\frac{x}{2}}.cos^2\frac{x}{2}$$

$$= 2\ tan\frac{x}{2}.\frac{1}{sec^2\frac{x}{2}} = \frac{2tan\frac{x}{2}}{1+tan^2\frac{x}{2}}$$

$$\therefore\ \boldsymbol{sinx = \frac{2z}{1+z^2}}$$

Finally, $\boldsymbol{x = 2\tan^{-1}z \Longrightarrow dx = \frac{2dz}{1+z^2}}$

Example5:

$$\int \frac{1}{1+cosx}\,dx =$$

Solution:

$$\int \frac{1}{1+cosx}\,dx = \int \frac{1+z^2}{2}\frac{2}{1+z^2}\,dz$$

$$= \int \mathrm{dz} = \mathrm{z} + \mathrm{C} = \tan\frac{\mathrm{x}}{2} + \mathrm{C}$$

Example6:

$$\int \frac{1}{2+sinx}dx =$$

Solution:

$$\int \frac{1}{2+sinx}dx = \int \frac{1+z^2}{1+2z+2z^2}\ \frac{2}{1+z^2}\ dz$$

$$= \int \frac{dz}{z^2+z+1} = \int \frac{dz}{\left(z+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \int \frac{1}{u^2+a^2}\ du = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\frac{2z+1}{\sqrt{3}} + C$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\frac{2tan\frac{x}{2}+1}{\sqrt{3}} + C$$

IP1.

$$\int \frac{\sqrt{9-w^2}}{w^2}\, dw =$$

Solution:

Step1:

To evaluate $\int \frac{\sqrt{9-w^2}}{w^2}\, dw$,

Put $w = 3sin\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$\Longrightarrow dw = 3cos\theta\, d\theta$

Step2:

$$\int \frac{\sqrt{9-w^2}}{w^2}\, dw$$

$$= \int \frac{\sqrt{9-(3sin\theta)^2}}{(3sin\theta)^2} 3cos\theta\, d\theta$$

$$= \int \frac{\sqrt{9-9sin^2\theta}}{9sin^2\theta} 3cos\theta\, d\theta$$

$$= \int \frac{\sqrt{1-sin^2\theta}}{sin^2\theta} cos\theta\, d\theta$$

$$= \int \frac{|cos\theta|}{sin^2\theta}\, cos\theta\, d\theta \qquad \left(\because \sqrt{cos^2\theta} = |cos\theta|\right)$$

$$= \int \frac{cos\theta}{sin^2\theta}\, cos\theta\, d\theta \left(\because cos\theta > 0, \quad for\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$$

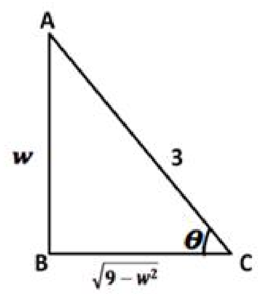
$$= \int cot^2\theta\ d\theta = \int (csc^2\theta - 1)\, d\theta$$

$$\int \frac{\sqrt{9-w^2}}{w^2}\, dw = -cot\theta - \theta + C \text{ .... } (1)$$

Step3:

We have $w = 3sin\theta$

$\Longrightarrow sin\theta = \frac{w}{3} \Longrightarrow \theta = \sin^{-1}\frac{w}{3}$



From the figure, $cot\theta = \frac{\sqrt{9-w^2}}{w}$

Step4:

$$(1) \Longrightarrow \int \frac{\sqrt{9-w^2}}{w^2}\, dw = -\frac{\sqrt{9-w^2}}{w} - \sin^{-1}\frac{w}{3} + C$$

P1.

$$\int \frac{8}{w^2\sqrt{4 - w^2}}\, dw =$$

Solution:

To evaluate $\int \frac{8}{w^2\sqrt{4-w^2}}\,dw$,

Put $w = 2sin\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$\Rightarrow dw = 2cos\theta\, d\theta$

$$\therefore \int \frac{8}{w^2\sqrt{4-w^2}}\,dw$$

$$= 8\int \frac{2cos\theta}{(2sin\theta)^2\sqrt{4-(2sin\theta)^2}}\,d\theta$$

$$= 16\int \frac{cos\theta}{8sin^2\theta\sqrt{1-sin^2\theta}}\,d\theta$$

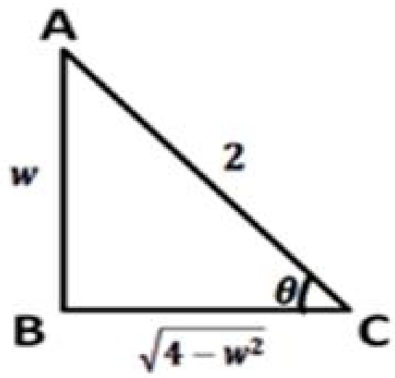
$$= 2\int \frac{cos\theta}{sin^2\theta|cos\theta|}\,d\theta \qquad (\because \sqrt{cos^2\theta} = |cos\theta|)$$

$$= 2\int \frac{cos\theta}{sin^2\theta|cos\theta|}\,d\theta \ \left(\because cos\theta > 0, for\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$$

$$\int \frac{8}{w^2\sqrt{4-w^2}}\,dw = 2\int csc^2\theta\ d\theta = -2cot\theta + C \quad .... (1)$$

We have $sin\theta = \frac{w}{2}$

From the figure, $cot\theta = \frac{\sqrt{4-w^2}}{w}$



$$(1) \Rightarrow \int \frac{8}{w^2\sqrt{4-w^2}}\,dw = -2cot\theta + C = \frac{-2\sqrt{4-w^2}}{w} + C$$

IP2.

$$\int \frac{x^3}{\sqrt{x^2+4}}\,dx =$$

Solution:

Step1:

To evaluate $\int \frac{x^3}{\sqrt{x^2+4}}\,dx$,

Put $x = 2tan\theta,\quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$\Rightarrow dx = 2sec^2\theta\ \ d\theta$

Step2:

$\therefore \int \frac{x^3}{\sqrt{x^2+4}}\,dx$

$$= \int \frac{(2tan\theta)^3}{\sqrt{4tan^2\theta+4}}\ 2sec^2\theta\ \ d\theta$$

$$= \int \frac{8tan^3\theta}{2\sqrt{sec^2\theta}} 2sec^2\theta\ \ d\theta$$

$$= 8\int \frac{tan^3\theta}{|sec\theta|} sec^2\theta\ \ d\theta \qquad \left(\because \sqrt{sec^2\theta} = |sec\theta|\right)$$

$$= 8\int \frac{tan^3\theta}{sec\theta} sec^2\theta\ \ d\theta \quad \left(\because sec\theta > 0\ \ for -\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$$

$$= 8\int tan^3\theta\, sec\theta\ \ d\theta = 8\int \frac{sin^3\theta}{cos^4\theta}\ d\theta$$

Put $cos\theta = t \Rightarrow\ -sin\theta\, d\theta = dt$

$$= 8\int \frac{(-sin\theta\,)(1-cos^2\theta)d\theta}{cos^4\theta}$$

$$= 8\int \frac{1-t^2}{t^4}\ dt$$

$$= 8\int \left(\frac{1}{t^2} - \frac{1}{t^4}\right) dt$$

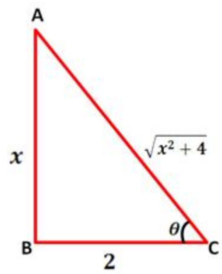
$$= 8\left(-\frac{1}{t} + \frac{1}{3t^3}\right) + C$$

$$= 8\left(-sec\theta + \frac{sec^3\theta}{3}\right) + C \text{ .... } (1)$$

Step3:

We have $x = 2tan\theta \Rightarrow tan\theta = \frac{x}{2}$

From figure, $sec\theta = \frac{\sqrt{x^2+4}}{2}$



Step4:

$$(1) \Rightarrow \int \frac{x^3}{\sqrt{x^2+4}}\,dx$$

$$= 8\left(-\frac{\sqrt{x^2+4}}{2}\right) + 8\frac{\left(\frac{\sqrt{x^2+4}}{2}\right)^3}{3} + C$$

$$= -4\sqrt{x^2+4} + \frac{1}{3}(x^2+4)^{\frac{3}{2}} + C$$

Hence, $\int \frac{x^3}{\sqrt{x^2+4}}\,dx = \frac{1}{3}(x^2+4)^{\frac{3}{2}} - 4\sqrt{x^2+4} + C$

P2.

$$\int \frac{dx}{(x^2+a^2)^2} = \cdots, \ a > 0$$

Solution:

To evaluate $\int \frac{dx}{(x^2+a^2)^2}$, $a > 0$,

Put $x = a\ tan\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$\Rightarrow dx = a\ \sec^2\theta\ d\theta$

$$\therefore \int \frac{dx}{(x^2+a^2)^2} = \int \frac{a\sec^2\theta}{\left((a^2\ tan^2\theta)+a^2\right)^2}\ d\theta$$

$$= \int \frac{a\sec^2\theta}{a^4(tan^2\theta+1)^2}\ d\theta$$

$$= \int \frac{\sec^2\theta}{a^3 sec^4\theta}\ d\theta$$

$$= \frac{1}{a^3}\int \cos^2\theta\ d\theta$$

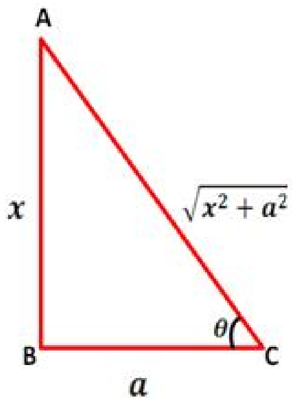
$$= \frac{1}{a^3}\int \left(\frac{1+cos2\theta}{2}\right)\ d\theta$$

$$= \frac{1}{2a^3}\int d\theta + \frac{1}{2a^3}\int \cos 2\theta\ d\theta$$

$$= \frac{1}{2a^3}\left[\theta + \frac{\sin 2\theta}{2}\right] \ldots\ldots\ldots\ldots\ldots (1)$$

We have $x = a\ tan\theta$,

$\Rightarrow tan\theta = \frac{x}{a}$, $\theta = tan^{-1}\left(\frac{x}{a}\right)$



From figure,

$sin\theta = \frac{x}{\sqrt{x^2+a^2}}$, $cos\theta = \frac{a}{\sqrt{a^2+x^2}}$

$$(1) \Rightarrow \int \frac{dx}{(x^2+a^2)^2} = \frac{1}{2a^3}\left[\theta + \frac{2\ sin\theta.cos\theta}{2}\right]$$

$$= \frac{1}{2a^3}\left[\tan^{-1}\left(\frac{x}{a}\right) + \frac{x}{\sqrt{x^2+a^2}}\frac{a}{\sqrt{x^2+a^2}}\right]$$

$$= \frac{1}{2a^3}\left[\tan^{-1}\left(\frac{x}{a}\right) + \frac{ax}{x^2+a^2}\right]$$

IP3.

$$\int \frac{\sqrt{y^2-25}}{y^3}\,dy, \qquad y>5$$

Solution:

Step1:

To evaluate $\int \frac{\sqrt{y^2-25}}{y^3}\,dy,\ \ y>5,$

Put $y = 5\,sec\theta\ ,\ 0<\theta<\frac{\pi}{2}$

$\Rightarrow dy = 5\,sec\theta.tan\theta\,d\theta$

Step2:

$$\int \frac{\sqrt{y^2-25}}{y^3}\,dy$$

$$= \int \frac{\sqrt{25\,sec^2\theta-25}}{(5\,sec\theta)^3}\,.5\,sec\theta tan\theta\,d\theta$$

$$= 5\int \frac{\sqrt{sec^2\theta-1}}{125\,sec^3\theta}\,.5\,sec\theta tan\theta\,d\theta$$

$$= \frac{1}{5}\int \frac{\sqrt{tan^2\theta}}{sec^2\theta}\,.tan\theta\,d\theta$$

$$= \frac{1}{5}\int \frac{|tan\theta|}{sec^2\theta}\,.tan\theta\,d\theta \quad \left(\because\ \sqrt{tan^2\theta} = |tan\theta|\right)$$

$$= \frac{1}{5}\int \frac{tan\theta}{sec^2\theta}\,.tan\theta\,d\theta \quad \left(\because\ tan\theta>0, for\ 0<\theta<\frac{\pi}{2}\right)$$

$$= \frac{1}{5}\int tan^2\theta\,.cos^2\theta\,d\theta$$

$$= \frac{1}{5}\int sin^2\theta\,d\theta$$

$$= \frac{1}{5}\int \left(\frac{1-cos2\theta}{2}\right)d\theta = \frac{1}{10}\left(\theta-\frac{sin2\theta}{2}\right)+c$$

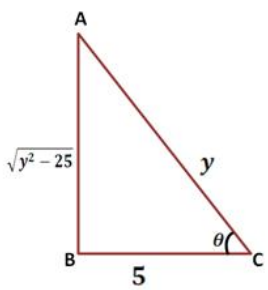
$$= \frac{1}{10}(\theta - sin\theta\,cos\theta) + C\ \ldots\ldots(1)$$

Step3:

We have $y = 5\,sec\theta$

$\Longrightarrow sec\theta = \frac{y}{5}\ ,\theta = sec^{-1}\left(\frac{y}{5}\right)$

$\Rightarrow sin\theta = \frac{\sqrt{y^2-25}}{y}, cos\theta = \frac{5}{y}$



Step 4:

$$\therefore (1) \Longrightarrow \int \frac{\sqrt{y^2-25}}{y}\,dy = \frac{1}{10}\left(\sec^{-1}\left(\frac{y}{5}\right) - \frac{\sqrt{y^2-25}}{y}.\frac{5}{y}\right)+C$$

$$= \frac{\sec^{-1}\left(\frac{y}{5}\right)}{10} - \frac{\sqrt{y^2-25}}{2y^2} + C$$

P3.

$$\int \frac{\sqrt{y^2 - 49}}{y}\, dy, \qquad y > 7$$

Solution:

To evaluate $\int \frac{\sqrt{y^2-49}}{7}\, dy \quad , \quad y > 7\,,$

Put $y = 7\, sec\theta,\ 0 < \theta < \frac{\pi}{2}$

$\Rightarrow dy = 7\, sec\theta . tan\theta\, d\theta$

$$\therefore \int \frac{\sqrt{y^2-49}}{7}\, dy = \int \frac{\sqrt{49sec^2\theta - sec^2\theta}}{7sec\theta}\, 7sec\theta . tan\theta\, d\theta$$

$$= 7\int \frac{\sqrt{sec^2\theta - 1}}{7sec\theta} tan\theta\, d\theta$$

$$= 7\int \sqrt{tan^2\theta} . tan\theta\, d\theta$$

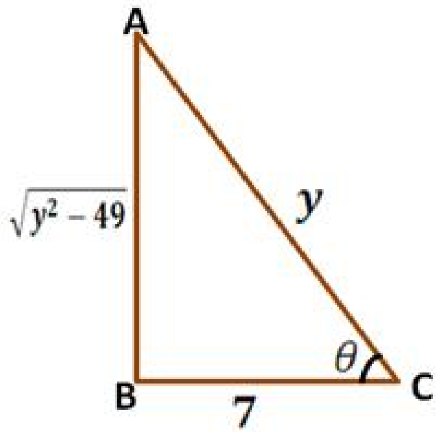
$$= 7\int |tan\theta| tan\theta\ d\theta \quad \left(\because \sqrt{tan^2\theta} = |tan\theta|\right)$$

$$= 7\int tan^2\theta\, d\theta \quad \left(\because tan\theta > 0\ \ for\ 0 < \theta < \frac{\pi}{2}\right)$$

$$= 7\int (sec^2\theta - 1) d\theta = 7(tan\theta - \theta) + C \ \ ......\ (1)$$

We have $y = 7sec\theta$

$\Rightarrow sec\theta = \frac{y}{7}, \quad \theta = \sec^{-1}\left(\frac{y}{7}\right)$



From figure, $tan\theta = \frac{\sqrt{y^2-49}}{7}$

$$(1) \Rightarrow \int \frac{\sqrt{y^2-49}}{7}\, dy = 7\left[\frac{\sqrt{y^2-49}}{7} - \sec^{-1}\left(\frac{y}{7}\right)\right] + C$$

IP4.

$$\int \frac{dx}{3cosx + 4sinx + 6} =$$

Solution:

Step1:

To evaluate $\int \frac{dx}{3cosx+4sinx+6}$,

Put $z = tan\frac{x}{2} \Rightarrow x = 2\tan^{-1} z$

$\therefore \quad sinx = \frac{2z}{1+z^2} \quad , \quad cosx = \frac{1-z^2}{1+z^2} \quad , \quad dx = \frac{2}{1+z^2}dz$

Step2:

$$\int \frac{dx}{3cosx + 4sinx + 6}$$

$$= \int \frac{\left(\frac{2}{1+z^2}\right)dz}{3\left(\frac{1-z^2}{1+z^2}\right)+4\left(\frac{2z}{1+z^2}\right)+6}$$

$$= \int \frac{1+z^2}{3(1-z^2)+8z+6(1+z^2)}\left(\frac{2}{1+z^2}\right)dz$$

$$= \int \frac{2dz}{3z^2+8z+9}$$

$$= \frac{2}{3}\int \frac{dz}{\left(z+\frac{4}{3}\right)^2-\frac{16}{9}+3}$$

$$= \frac{2}{3}\int \frac{dz}{\left(z+\frac{4}{3}\right)^2+\left(\frac{\sqrt{11}}{3}\right)^2}$$

$$= \frac{2}{3}.\frac{3}{\sqrt{11}}\tan^{-1}\left[\frac{z+\frac{4}{3}}{\frac{\sqrt{11}}{3}}\right] + C$$

$$= \frac{2}{\sqrt{11}}\tan^{-1}\left(\frac{3z+4}{\sqrt{11}}\right) + C$$

$$= \frac{2}{\sqrt{11}}\tan^{-1}\left(\frac{3tan\frac{x}{2}+4}{\sqrt{11}}\right) + C$$

Hence, $\int \frac{dx}{3cosx+4sinx+6} = \frac{2}{\sqrt{11}}\tan^{-1}\left(\frac{3tan\frac{x}{2}+4}{\sqrt{11}}\right) + C$

P4.

$$\int \frac{dx}{1 + sinx + cosx} =$$

Solution:

To evaluate $\int \frac{dx}{1+sinx+cosx}$,

Put $z = tan\frac{x}{2} \Longrightarrow x = 2\tan^{-1} z$

$\therefore \quad sinx = \frac{2z}{1+z^2} \quad , \quad cosx = \frac{1-z^2}{1+z^2} \quad , \quad dx = \frac{2}{1+z^2}dz$

Now, $\int \frac{dx}{1+sinx+cosx}$

$$= \int \frac{\left(\frac{2}{1+z^2}\right)dz}{1+\left(\frac{2z}{1+z^2}\right)+\left(\frac{1-z^2}{1+z^2}\right)}$$

$$= \int \frac{1+z^2}{(1+z^2)+2z+(1-z^2)}\left(\frac{2}{1+z^2}\right)dz$$

$$= \int \frac{2}{1+2z}dz$$

$$= 2\frac{\log(1+2z)}{2} + C$$

$$= \log\left(2tan\frac{x}{2} + 1\right) + C$$

1. Evaluate

a. $\int \frac{x^2}{\sqrt{1-x^2}}\,dx$

b. $\int \sqrt{25-t^2}\;dt$

c. $\int_0^{\frac{\sqrt{3}}{2}} \frac{4x^2}{(1-x^2)^{\frac{3}{2}}}\;dx$

d. $\int_0^1 \frac{dx}{(4-x^2)^{\frac{3}{2}}}\;dx$

e. $\int \frac{v^2}{(1-v^2)^{\frac{5}{2}}}\;dv$

f. $\int \frac{(1-r^2)^{\frac{5}{2}}}{r^8}\;dr$

g. $\int \frac{(1-x^2)^{\frac{3}{2}}}{x^6}\,dx$

2. Evaluate

a. $\int \frac{dx}{x^2\sqrt{x^2-1}}$

b. $\int \frac{2dx}{x^3\sqrt{x^2-1}}$

c. $\int \frac{x^2}{(x^2-1)^{\frac{5}{2}}}\,dx$

d. $\int \frac{5dx}{\sqrt{25x^2-9}}$

e. $\int \frac{dx}{(x^2-1)^{\frac{3}{2}}}$

f. $\int \frac{dx}{\sqrt{x^2-1}}$

3. Evaluate

a. $\int \frac{x^3\,dx}{\sqrt{x^2+4}}$

b. $\int \frac{dx}{x^2\sqrt{x^2+1}}$

c. $\int \frac{8dx}{(4x^2+1)^2}$

d. $\int \frac{6dt}{(9t^2+1)^2}$

e. $\int_{1/12}^{1/4} \frac{2dt}{\sqrt{t}+4t\sqrt{t}}$

f. $\int_{0}^{\ln 4} \frac{e^t}{\sqrt{e^{2t}+9}}\,dt$

4. Evaluate

a. $\int \frac{dx}{5+4\cos x}$

b. $\int \frac{dx}{4+5\sin x}$

c. $\int \frac{dx}{2-3\cos 2x}$

d. $\int \frac{dx}{4\cos x+3\sin x}$

e. $\int \frac{dx}{\sin x+\sqrt{3}\cos x}$

f. $\int \frac{8dx}{(4x^2+1)^2}$

12.6

Integral Tables

Learning objectives:

In this module, we study

- To evaluate a given integral with the help of the integral tables by transforming it into an integral listed in the integral table.
- To explain the use of integral table to evaluate certain integrals

  AND

- To practice the related problems.

The basic techniques of integration are substitution and integration by parts. We apply these techniques to transform unfamiliar integrals into integrals whose forms we recognize or can find in a table. The integrals in the tables come from applying substitutions and integration by parts.

When an integral matches an integral in the table or can be changed into one of the tabulated integrals with some appropriate combination of algebra, trigonometry, substitution, and calculus, we have a ready-made solution for the problem at hand. The examples in this module show how the formulas in integral tables are used. *The emphasis is on use.*

The integration formulas, given in the appendix, are stated in terms of constants a, b, c, m, n, and so on. These constants can usually assume any real value and need not be integers. The formulas also assume that the constants do not take on values that require dividing by zero or taking even roots of negative numbers.

Example 1

Find $\int x(2x+5)^{-1}\,dx$

Solution

We use formula 8

$$\int x(ax+b)^{-1}dx = \frac{x}{a} - \frac{b}{a^2}\ln|ax+b| + C$$

With $a = 2$ and $b = 5$, we have

$$\int x(2x+5)^{-1}dx = \frac{x}{2} - \frac{5}{4}\ln|2x+5| + C$$

Example 2

Find $\int \frac{dx}{x\sqrt{2x+4}}$

Solution

We use formula 13(b):

$$\int \frac{dx}{x\sqrt{ax+b}} = \frac{1}{\sqrt{b}}\ln\left|\frac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}}\right| + C \quad \text{if} \quad b>0$$

With $a = 2$ and $b = 4$, we have

$$\int \frac{dx}{x\sqrt{2x+4}} = \frac{1}{\sqrt{4}}\ln\left|\frac{\sqrt{2x+4}-\sqrt{4}}{\sqrt{2x+4}+\sqrt{4}}\right| + C$$
$$= \frac{1}{2}\ln\left|\frac{\sqrt{2x+4}-2}{\sqrt{2x+4}+2}\right| + C$$

Example 3

Find $\int \frac{dx}{x\sqrt{2x-4}}$

Solution

We use formula 13(a):

$$\int \frac{dx}{x\sqrt{ax-b}} = \frac{2}{\sqrt{b}}\tan^{-1}\sqrt{\frac{ax-b}{b}} + C$$

With $a = 2$ and $b = 4$, we have

$$\int \frac{dx}{x\sqrt{2x+4}} = \frac{2}{\sqrt{4}}\tan^{-1}\sqrt{\frac{2x-4}{4}} + C$$
$$= \tan^{-1}\sqrt{\frac{x-2}{2}} + C$$

Example 4

Find $\int \frac{dx}{x^2\sqrt{2x-4}}$

Solution

We begin with formula 15:

$$\int \frac{dx}{x^2\sqrt{ax+b}} = -\frac{\sqrt{ax+b}}{bx} - \frac{a}{2b}\int \frac{dx}{x\sqrt{ax+b}} + C$$

With $a = 2$ and $b = -4$, we have

$$\int \frac{dx}{x^2\sqrt{2x-4}} = -\frac{\sqrt{2x-4}}{-4x} + \frac{2}{2\cdot 4}\int \frac{dx}{x\sqrt{2x-4}} + C$$

We then use formula 13(a) to evaluate the integral on the right (Example 3) to obtain

$$\int \frac{dx}{x^2\sqrt{2x-4}} = \frac{\sqrt{2x-4}}{4x} + \frac{1}{4}\tan^{-1}\sqrt{\frac{x-2}{2}} + C$$

Example 5

Find $\int x\sin^{-1}x\,dx$

Solution

We use formula 99:

$$\int x^n\sin^{-1}ax\,dx = \frac{x^{n+1}}{n+1}\sin^{-1}ax - \frac{a}{n+1}\int \frac{x^{n+1}dx}{\sqrt{1-a^2x^2}}, \quad n \neq -1$$

With $n = 1$ and $a = 1$, we have

$$\int x\sin^{-1}x\,dx = \frac{x^2}{2}\sin^{-1}x - \frac{1}{2}\int \frac{x^2dx}{\sqrt{1-x^2}}$$

The integral on the right is found in the table as formula 33:

$$\int \frac{x^2}{\sqrt{a^2-x^2}}dx = \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) - \frac{1}{2}x\sqrt{a^2-x^2} + C$$

With $a = 1$,

$$\int \frac{x^2}{\sqrt{1-x^2}}dx = \frac{1}{2}\sin^{-1}x - \frac{1}{2}x\sqrt{1-x^2} + C$$

The combined result is

$$\int x\sin^{-1}x\,dx = \frac{x^2}{2}\sin^{-1}x - \frac{1}{2}\left(\frac{1}{2}\sin^{-1}x - \frac{1}{2}x\sqrt{1-x^2}\right) + C'$$
$$= \left(\frac{x^2}{2} - \frac{1}{4}\right)\sin^{-1}x + \frac{1}{4}x\sqrt{1-x^2} + C'$$

There are powerful programs, known as Computer Algebraic Systems (CAS), which integrate many indefinite integrals symbolically. Some of the well-known programs are ***Maple*** and ***Mathematica***. The integration is performed with the **integrate command** specified by the particular system (*int* in Maple, *Integrate* in Mathematica).

The computer algebraic systems are generally faster than tables and usually they do not require you to rewrite integrals in special recognizable forms first.

We will learn the computer algebraic systems in a later course.

Appendix

1. $\int u\,dv = uv - \int v\,du$
2. $\int a^u du = \frac{a^u}{\ln a} + C, \quad a \neq 1, \quad a > 0$
3. $\int \cos u\,du = \sin u + C$
4. $\int \sin u\,du = -\cos u + C$
5. $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C, \quad n \neq -1$
6. $\int (ax+b)^{-1}dx = \frac{1}{a}\ln|ax+b| + C$
7. $\int x(ax+b)^n dx = \frac{(ax+b)^{n+1}}{a^2}\left[\frac{ax+b}{n+2} - \frac{b}{n+1}\right] + C, \quad n \neq -1, -2$
8. $\int x(ax+b)^{-1}dx = \frac{x}{a} - \frac{b}{a^2}\ln|ax+b| + C$
9. $\int x(ax+b)^{-2}dx = \frac{1}{a^2}\left[\ln|ax+b| + \frac{b}{ax+b}\right] + C$
10. $\int \frac{dx}{x(ax+b)} = \frac{1}{b}\ln\left|\frac{x}{ax+b}\right| + C$
11. $\int (\sqrt{ax+b})^n dx = \frac{2}{a}\frac{(\sqrt{ax+b})^{n+2}}{n+2} + C, \quad n \neq -2$
12. $\int \frac{\sqrt{ax+b}}{x}dx = 2\sqrt{ax+b} + b\int \frac{dx}{x\sqrt{ax+b}}$
13. a) $\int \frac{dx}{x\sqrt{ax-b}} = \frac{2}{\sqrt{b}}\tan^{-1}\sqrt{\frac{ax-b}{b}} + C$

    b) $\int \frac{dx}{x\sqrt{ax+b}} = \frac{1}{\sqrt{b}}\ln\left|\frac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}}\right| + C$
14. $\int \frac{\sqrt{ax+b}}{x^2}dx = -\frac{\sqrt{ax+b}}{x} + \frac{a}{2}\int \frac{dx}{x\sqrt{ax+b}} + C$
15. $\int \frac{dx}{x^2\sqrt{ax+b}} = -\frac{\sqrt{ax+b}}{bx} - \frac{a}{2b}\int \frac{dx}{x\sqrt{ax+b}} + C$
16. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$
17. $\int \frac{dx}{(a^2+x^2)^2} = \frac{x}{2a^2(a^2+x^2)} + \frac{1}{2a^3}\tan^{-1}\frac{x}{a} + C$
18. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right| + C$
19. $\int \frac{dx}{(a^2-x^2)^2} = \frac{x}{2a^2(a^2-x^2)} + \frac{1}{4a^3}\ln\left|\frac{x+a}{x-a}\right| + C$
20. $\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\frac{x}{a} + C = \ln(x+\sqrt{a^2+x^2}) + C$
21. $\int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\ln(x+\sqrt{a^2+x^2}) + C$
22. $\int x^2\sqrt{a^2+x^2}\,dx = \frac{x}{8}(a^2+2x^2)\sqrt{a^2+x^2} - \frac{a^4}{8}\ln(x+\sqrt{a^2+x^2}) + C$
23. $\int \frac{\sqrt{a^2+x^2}}{x}dx = \sqrt{a^2+x^2} - a\ln\left|\frac{a+\sqrt{a^2+x^2}}{x}\right| + C$
24. $\int \frac{\sqrt{a^2+x^2}}{x^2}dx = \ln(x+\sqrt{a^2+x^2}) - \frac{\sqrt{a^2+x^2}}{x} + C$
25. $\int \frac{x^2}{\sqrt{a^2+x^2}}dx = -\frac{a^2}{2}\ln(x+\sqrt{a^2+x^2}) + \frac{x\sqrt{a^2+x^2}}{2} + C$
26. $\int \frac{dx}{x\sqrt{a^2+x^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2+x^2}}{x}\right| + C$
27. $\int \frac{dx}{x^2\sqrt{a^2+x^2}} = -\frac{\sqrt{a^2+x^2}}{a^2x} + C$
28. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$
29. $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$
30. $\int x^2\sqrt{a^2-x^2}\,dx = \frac{a^4}{8}\sin^{-1}\frac{x}{a} - \frac{1}{8}x\sqrt{a^2-x^2}(a^2-2x^2) + C$
31. $\int \frac{\sqrt{a^2-x^2}}{x}dx = \sqrt{a^2-x^2} - a\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right| + C$
32. $\int \frac{\sqrt{a^2-x^2}}{x^2}dx = -\sin^{-1}\frac{x}{a} - \frac{\sqrt{a^2-x^2}}{x} + C$
33. $\int \frac{x^2}{\sqrt{a^2-x^2}}dx = \frac{a^2}{2}\sin^{-1}\frac{x}{a} - \frac{1}{2}x\sqrt{a^2-x^2} + C$
34. $\int \frac{dx}{x\sqrt{a^2-x^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right| + C$
35. $\int \frac{dx}{x^2\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{a^2x} + C$
36. $\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a} + C = \ln\left|x+\sqrt{x^2-a^2}\right| + C$
37. $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + C$
38. $\int (\sqrt{x^2-a^2})^n dx = \frac{x(\sqrt{x^2-a^2})^n}{n+1} - \frac{na^2}{n+1}\int (\sqrt{x^2-a^2})^{n-2}dx, \quad n \neq -1$
39. $\int \frac{dx}{(\sqrt{x^2-a^2})^n} = \frac{x(\sqrt{x^2-a^2})^{2-n}}{(2-n)a^2} - \frac{n-3}{(n-2)a^2}\int \frac{dx}{(\sqrt{x^2-a^2})^{n-2}}, \quad n \neq 2$
40. $\int x(\sqrt{x^2-a^2})^n dx = \frac{(\sqrt{x^2-a^2})^{n+2}}{n+2} + C, \quad n \neq -2$
41. $\int x^2\sqrt{x^2-a^2}\,dx = \frac{x}{8}(2x^2-a^2)\sqrt{x^2-a^2} - \frac{a^4}{8}\ln\left|x+\sqrt{x^2-a^2}\right| + C$
42. $\int \frac{\sqrt{x^2-a^2}}{x}dx = \sqrt{x^2-a^2} - a\sec^{-1}\left|\frac{x}{a}\right| + C$
43. $\int \frac{\sqrt{x^2-a^2}}{x^2}dx = \ln\left|x+\sqrt{x^2-a^2}\right| - \frac{\sqrt{x^2-a^2}}{x} + C$
44. $\int \frac{x^2}{\sqrt{x^2-a^2}}dx = \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + \frac{x}{2}\sqrt{x^2-a^2} + C$
45. $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C = \frac{1}{a}\cos^{-1}\left|\frac{a}{x}\right| + C$
46. $\int \frac{dx}{x^2\sqrt{x^2-a^2}} = \frac{\sqrt{x^2-a^2}}{a^2x} + C$
47. $\int \frac{dx}{\sqrt{2ax-x^2}} = \sin^{-1}\left(\frac{x-a}{a}\right) + C$
48. $\int \sqrt{2ax-x^2}\,dx = \frac{x-a}{2}\sqrt{2ax-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right) + C$
49. $\int (\sqrt{2ax-x^2})^n dx = \frac{(x-a)(\sqrt{2ax-x^2})^n}{n+1} + \frac{na^2}{n+1}\int (\sqrt{2ax-x^2})^{n-2}dx$
50. $\int \frac{dx}{(\sqrt{2ax-x^2})^n} = \frac{(x-a)(\sqrt{2ax-x^2})^{2-n}}{(n-2)a^2} + \frac{n-3}{(n-2)a^2}\int \frac{dx}{(\sqrt{2ax-x^2})^{n-2}}$
51. $\int x\sqrt{2ax-x^2}\,dx = \frac{(x+a)(2x-3a)\sqrt{2ax-x^2}}{6} + \frac{a^3}{2}\sin^{-1}\left(\frac{x-a}{a}\right) + C$
52. $\int \frac{\sqrt{2ax-x^2}}{x}dx = \sqrt{2ax-x^2} + a\sin^{-1}\left(\frac{x-a}{a}\right) + C$
53. $\int \frac{\sqrt{2ax-x^2}}{x^2}dx = -2\sqrt{\frac{2a-x}{x}} - \sin^{-1}\left(\frac{x-a}{a}\right) + C$
54. $\int \frac{x\,dx}{\sqrt{2ax-x^2}} = a\sin^{-1}\left(\frac{x-a}{a}\right) - \sqrt{2ax-x^2} + C$
55. $\int \frac{dx}{x\sqrt{2ax-x^2}} = -\frac{1}{a}\sqrt{\frac{2a-x}{x}} + C$
56. $\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$
57. $\int \cos ax\,dx = \frac{1}{a}\sin ax + C$
58. $\int \sin^2 ax\,dx = \frac{x}{2} - \frac{\sin 2ax}{4a} + C$
59. $\int \cos^2 ax\,dx = \frac{x}{2} + \frac{\sin 2ax}{4a} + C$
60. $\int \sin^n ax\,dx = -\frac{\sin^{n-1}ax\cos ax}{na} + \frac{n-1}{n}\int \sin^{n-2}ax\,dx$
61. $\int \cos^n ax\,dx = \frac{\cos^{n-1}ax\sin ax}{na} + \frac{n-1}{n}\int \cos^{n-2}ax\,dx$
62. a) $\int \sin ax\cos bx\,dx = -\frac{\cos(a+b)x}{2(a+b)} - \frac{\cos(a-b)x}{2(a-b)} + C, \quad a^2 \neq b^2$

    b) $\int \sin ax\sin bx\,dx = \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)} + C, \quad a^2 \neq b^2$

    c) $\int \cos ax\cos bx\,dx = \frac{\sin(a-b)x}{2(a-b)} + \frac{\sin(a+b)x}{2(a+b)} + C, \quad a^2 \neq b^2$
63. $\int \sin ax\cos ax\,dx = -\frac{\cos 2ax}{4a} + C$
64. $\int \sin^n ax\cos ax\,dx = \frac{\sin^{n+1}ax}{(n+1)a} + C, \quad n \neq -1$
65. $\int \frac{\cos ax}{\sin ax}dx = \frac{1}{a}\ln|\sin ax| + C$
66. $\int \cos^n ax\sin ax\,dx = -\frac{\cos^{n+1}ax}{(n+1)a} + C, \quad n \neq -1$
67. $\int \frac{\sin ax}{\cos ax}dx = -\frac{1}{a}\ln|\cos ax| + C$
68. $\int \sin^n ax\cos^m ax\,dx = -\frac{\sin^{n-1}ax\cos^{m+1}ax}{a(m+n)} + \frac{n-1}{m+n}\int \sin^{n-2}ax\cos^m ax\,dx, \quad n \neq -m$ (reduces $\sin^n ax$)
69. $\int \sin^n ax\cos^m ax\,dx = \frac{\sin^{n+1}ax\cos^{m-1}ax}{a(m+n)} + \frac{m-1}{m+n}\int \sin^n ax\cos^{m-2}ax\,dx, \quad m \neq -n$ (reduces $\cos^m ax$)
70. $\int \frac{dx}{b+c\sin ax} = \frac{-2}{a\sqrt{b^2-c^2}}\tan^{-1}\left[\sqrt{\frac{b-c}{b+c}}\tan\left(\frac{\pi}{4}-\frac{ax}{2}\right)\right] + C, \quad b^2 > c^2$
71. $\int \frac{dx}{b+c\sin ax} = \frac{-1}{a\sqrt{c^2-b^2}}\ln\left|\frac{c+b\sin ax+\sqrt{c^2-b^2}\cos ax}{b+c\sin ax}\right| + C, \quad b^2 < c^2$
72. $\int \frac{dx}{1+\sin ax} = -\frac{1}{a}\tan\left(\frac{\pi}{4}-\frac{ax}{2}\right) + C$
73. $\int \frac{dx}{1-\sin ax} = \frac{1}{a}\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right) + C$
74. $\int \frac{dx}{b+c\cos ax} = \frac{2}{a\sqrt{b^2-c^2}}\tan^{-1}\left[\sqrt{\frac{b-c}{b+c}}\tan\frac{ax}{2}\right] + C, \quad b^2 > c^2$
75. $\int \frac{dx}{b+c\cos ax} = \frac{1}{a\sqrt{c^2-b^2}}\ln\left|\frac{c+b\cos ax+\sqrt{c^2-b^2}\sin ax}{b+c\cos ax}\right| + C, \quad b^2 < c^2$
76. $\int \frac{dx}{1+\cos ax} = \frac{1}{a}\tan\frac{ax}{2} + C$
77. $\int \frac{dx}{1-\cos ax} = -\frac{1}{a}\cot\frac{ax}{2} + C$
78. $\int x\sin ax\,dx = \frac{1}{a^2}\sin ax - \frac{x}{a}\cos ax + C$
79. $\int x\cos ax\,dx = \frac{1}{a^2}\cos ax + \frac{x}{a}\sin ax + C$
80. $\int x^n\sin ax\,dx = -\frac{x^n}{a}\cos ax + \frac{n}{a}\int x^{n-1}\cos ax\,dx$
81. $\int x^n\cos ax\,dx = \frac{x^n}{a}\sin ax - \frac{n}{a}\int x^{n-1}\sin ax\,dx$
82. $\int \tan ax\,dx = \frac{1}{a}\ln|\sec ax| + C$
83. $\int \cot ax\,dx = \frac{1}{a}\ln|\sin ax| + C$
84. $\int \tan^2 ax\,dx = \frac{1}{a}\tan ax - x + C$
85. $\int \cot^2 ax\,dx = -\frac{1}{a}\cot ax - x + C$
86. $\int \tan^n ax\,dx = \frac{\tan^{n-1}ax}{a(n-1)} - \int \tan^{n-2}ax\,dx, \quad n \neq 1$
87. $\int \cot^n ax\,dx = -\frac{\cot^{n-1}ax}{a(n-1)} - \int \cot^{n-2}ax\,dx, \quad n \neq 1$
88. $\int \sec ax\,dx = \frac{1}{a}\ln|\sec ax + \tan ax| + C$
89. $\int \csc ax\,dx = -\frac{1}{a}\ln|\csc ax + \cot ax| + C$
90. $\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C$
91. $\int \csc^2 ax\,dx = -\frac{1}{a}\cot ax + C$
92. $\int \sec^n ax\,dx = \frac{\sec^{n-2}ax\tan ax}{a(n-1)} + \frac{n-2}{n-1}\int \sec^{n-2}ax\,dx, \quad n \neq 1$
93. $\int \csc^n ax\,dx = -\frac{\csc^{n-2}ax\cot ax}{a(n-1)} + \frac{n-2}{n-1}\int \csc^{n-2}ax\,dx, \quad n \neq 1$
94. $\int \sec^n ax\tan ax\,dx = \frac{\sec^n ax}{na} + C, \quad n \neq 0$
95. $\int \csc^n ax\cot ax\,dx = -\frac{\csc^n ax}{na} + C, \quad n \neq 0$
96. $\int \sin^{-1}ax\,dx = x\sin^{-1}ax + \frac{1}{a}\sqrt{1-a^2x^2} + C$
97. $\int \cos^{-1}ax\,dx = x\cos^{-1}ax - \frac{1}{a}\sqrt{1-a^2x^2} + C$
98. $\int \tan^{-1}ax\,dx = x\tan^{-1}ax - \frac{1}{2a}\ln(1+a^2x^2) + C$
99. $\int x^n\sin^{-1}ax\,dx = \frac{x^{n+1}}{n+1}\sin^{-1}ax - \frac{a}{n+1}\int \frac{x^{n+1}dx}{\sqrt{1-a^2x^2}}, \quad n \neq -1$
100. $\int x^n\cos^{-1}ax\,dx = \frac{x^{n+1}}{n+1}\cos^{-1}ax + \frac{a}{n+1}\int \frac{x^{n+1}dx}{\sqrt{1-a^2x^2}}, \quad n \neq -1$
101. $\int x^n\tan^{-1}ax\,dx = \frac{x^{n+1}}{n+1}\tan^{-1}ax - \frac{a}{n+1}\int \frac{x^{n+1}dx}{1+a^2x^2}, \quad n \neq -1$
102. $\int e^{ax}dx = \frac{1}{a}e^{ax} + C$
103. $\int b^{ax}dx = \frac{1}{a}\frac{b^{ax}}{\ln b} + C, \quad b > 0, \ b \neq 1$
104. $\int xe^{ax}dx = \frac{e^{ax}}{a^2}(ax-1) + C$
105. $\int x^ne^{ax}dx = \frac{1}{a}x^ne^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}dx$
106. $\int x^nb^{ax}dx = \frac{x^nb^{ax}}{a\ln b} - \frac{n}{a\ln b}\int x^{n-1}b^{ax}dx, \quad b > 0, b \neq 1$
107. $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + C$
108. $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + C$
109. $\int \ln ax\,dx = x\ln ax - x + C$
110. $\int x^n(\ln ax)^m dx = \frac{x^{n+1}(\ln ax)^m}{n+1} - \frac{m}{n+1}\int x^n(\ln ax)^{m-1}dx, \quad n \neq -1$
111. $\int x^{-1}(\ln ax)^m dx = \frac{(\ln ax)^{m+1}}{m+1} + C, \quad m \neq -1$
112. $\int \frac{dx}{x\ln ax} = \ln|\ln ax| + C$
113. $\int \sinh ax\,dx = \frac{1}{a}\cosh ax + C$
114. $\int \cosh ax\,dx = \frac{1}{a}\sinh ax + C$
115. $\int \sinh^2 ax\,dx = \frac{\sinh 2ax}{4a} - \frac{x}{2} + C$
116. $\int \cosh^2 ax\,dx = \frac{\sinh 2ax}{4a} + \frac{x}{2} + C$
117. $\int \sinh^n ax\,dx = \frac{\sinh^{n-1}ax\cosh ax}{na} - \frac{n-1}{n}\int \sinh^{n-2}ax\,dx, \quad n \neq 0$
118. $\int \cosh^n ax\,dx = \frac{\cosh^{n-1}ax\sinh ax}{na} + \frac{n-1}{n}\int \cosh^{n-2}ax\,dx, \quad n \neq 0$
119. $\int x\sinh ax\,dx = \frac{x}{a}\cosh ax - \frac{1}{a^2}\sinh ax + C$
120. $\int x\cosh ax\,dx = \frac{x}{a}\sinh ax - \frac{1}{a^2}\cosh ax + C$
121. $\int x^n\sinh ax\,dx = \frac{x^n}{a}\cosh ax - \frac{n}{a}\int x^{n-1}\cosh ax\,dx$
122. $\int x^n\cosh ax\,dx = \frac{x^n}{a}\sinh ax - \frac{n}{a}\int x^{n-1}\sinh ax\,dx$
123. $\int \tanh ax\,dx = \frac{1}{a}\ln(\cosh ax) + C$
124. $\int \coth ax\,dx = \frac{1}{a}\ln|\sinh ax| + C$
125. $\int \tanh^2 ax\,dx = x - \frac{1}{a}\tanh ax + C$
126. $\int \coth^2 ax\,dx = x - \frac{1}{a}\coth ax + C$
127. $\int \tanh^n ax\,dx = -\frac{\tanh^{n-1}ax}{(n-1)a} + \int \tanh^{n-2}ax\,dx, \quad n \neq 1$
128. $\int \coth^n ax\,dx = -\frac{\coth^{n-1}ax}{(n-1)a} + \int \coth^{n-2}ax\,dx, \quad n \neq 1$
129. $\int \operatorname{sech} ax\,dx = \frac{1}{a}\sin^{-1}(\tanh ax) + C$
130. $\int \operatorname{csch} ax\,dx = \frac{1}{a}\ln\left|\tanh\frac{ax}{2}\right| + C$
131. $\int \operatorname{sech}^2 ax\,dx = \frac{1}{a}\tanh ax + C$
132. $\int \operatorname{csch}^2 ax\,dx = -\frac{1}{a}\coth ax + C$
133. $\int \operatorname{sech}^n ax\,dx = \frac{\operatorname{sech}^{n-2}ax\tanh ax}{(n-1)a} + \frac{n-2}{n-1}\int \operatorname{sech}^{n-2}ax\,dx, \quad n \neq 1$
134. $\int \operatorname{csch}^n ax\,dx = -\frac{\operatorname{csch}^{n-2}ax\coth ax}{(n-1)a} - \frac{n-2}{n-1}\int \operatorname{csch}^{n-2}ax\,dx, \quad n \neq 1$
135. $\int \operatorname{sech}^n ax\tanh ax\,dx = -\frac{\operatorname{sech}^n ax}{na} + C, \quad n \neq 0$
136. $\int \operatorname{csch}^n ax\coth ax\,dx = -\frac{\operatorname{csch}^n ax}{na} + C, \quad n \neq 0$
137. $\int e^{ax}\sinh bx\,dx = \frac{e^{ax}}{2}\left[\frac{e^{bx}}{a+b} - \frac{e^{-bx}}{a-b}\right] + C, \quad a^2 \neq b^2$
138. $\int e^{ax}\cosh bx\,dx = \frac{e^{ax}}{2}\left[\frac{e^{bx}}{a+b} + \frac{e^{-bx}}{a-b}\right] + C, \quad a^2 \neq b^2$
139. $\int_0^\infty x^{n-1}e^{-x}dx = \Gamma(n) = (n-1)!, \quad n > 0$
140. $\int_0^\infty e^{-ax^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}, \quad a > 0$
141. $\int_0^{\pi/2}\sin^n x\,dx = \int_0^{\pi/2}\cos^n x\,dx = \begin{cases} \frac{1\cdot 3\cdot 5\cdots(n-1)}{2\cdot 4\cdot 6\cdots n}\cdot\frac{\pi}{2}, & \text{if } n \text{ is an even integer} \geq 2 \\ \frac{2\cdot 4\cdot 6\cdots(n-1)}{3\cdot 5\cdot 7\cdots n}, & \text{if } n \text{ is an odd integer} \geq 3 \end{cases}$

IP1.

Prove that

$$a.\ \int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+C$$

where C is an arbitrary constant.

$$b.\ \int\sqrt{x^2-a^2}\,dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\cosh^{-1}\left(\frac{x}{a}\right)+C$$

where C is an arbitrary constant.

Proof:

a.

Step1:

To evaluate $\int\sqrt{a^2-x^2}\,dx$,

Put $x=a\,sin\theta,\ \theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

$\Longrightarrow dx=a\,cos\theta\,d\theta$

Step2:

$$\therefore\int\sqrt{a^2-x^2}\,dx=\int\sqrt{a^2-(a\,sin\theta)^2}\ .a\,cos\theta d\theta$$

$$=a\int\sqrt{1-sin^2\theta}\,.a\,cos\theta d\theta$$

$$=a^2\int cos^2\theta\,d\theta$$

$$=a^2\int\left(\frac{1+cos2\theta}{2}\right)d\theta$$

$$=\frac{a^2}{2}\left[\int d\theta+\int cos2\theta\,d\theta\right]$$

$$=\frac{a^2}{2}\left[\theta+\frac{sin2\theta}{2}\right]+C$$

where C is an arbitrary constant.

$$=\frac{a^2}{2}[\theta+sin\theta.cos\theta]+C\text{.........}\ (1)$$

Step3:

We have $sin\theta=\frac{x}{a}\Longrightarrow\theta=\sin^{-1}\left(\frac{x}{a}\right)$ $(1)\Longrightarrow$

$\int\sqrt{a^2-x^2}\,dx$

$$=\frac{a^2}{2}\left[\theta+sin\theta.\sqrt{1-sin^2\theta}\right]+C$$

$$=\frac{a^2}{2}\left[\sin^{-1}\left(\frac{x}{a}\right)+\frac{x}{a}\sqrt{1-\left(\frac{x}{a}\right)^2}\right]+C$$

$$=\frac{a^2}{2}\left[\sin^{-1}\left(\frac{x}{a}\right)+\frac{x}{a^2}\sqrt{a^2-x^2}\right]+C$$

$$=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+C$$

b.

Step1:

To evaluate $\int\sqrt{x^2-a^2}\,dx$,

Put $x=a\,cosh\theta$, $\theta\in[0,\infty)$

$\Longrightarrow dx=a\,sinh\theta\,d\theta$

Step2:

$$\therefore\int\sqrt{x^2-a^2}\,dx=\int\sqrt{(a\,cosh\theta)^2-a^2}\,.a\,sinh\theta\,d\theta$$

$$=a\int\sqrt{cosh^2\theta-1}\,.a\,sinh\theta\,d\theta$$

$$=a^2\int sinh^2\theta\,d\theta$$

$$=a^2\int\left(\frac{cosh2\theta-1}{2}\right)d\theta$$

$$=\frac{a^2}{2}\left[\int cosh2\theta\,d\theta-\int d\theta\right]$$

$$=\frac{a^2}{2}\left[\frac{sinh2\theta}{2}-\theta\right]+C$$

where C is an arbitrary constant.

$$=\frac{a^2}{2}[sinh\theta.cosh\theta-\theta]+C\text{.........}\ (1)$$

Step3:

We have $cosh\theta=\frac{x}{a}\Longrightarrow\theta=\cosh^{-1}\left(\frac{x}{a}\right)$

$(1)\Longrightarrow\int\sqrt{x^2-a^2}\,dx$

$$=\frac{a^2}{2}\left[cosh\theta.\sqrt{cosh^2\theta-1}-\theta\right]+C$$

$$=\frac{a^2}{2}\left[\frac{x}{a}\sqrt{\left(\frac{x}{a}\right)^2-1}-\cosh^{-1}\left(\frac{x}{a}\right)\right]+C$$

$$=\frac{a^2}{2}\left[\frac{x}{a^2}\sqrt{x^2-a^2}-\cosh^{-1}\left(\frac{x}{a}\right)\right]+C$$

$$=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\cosh^{-1}\left(\frac{x}{a}\right)+C$$

P1.

$$\int \sqrt{3 + 8x - 3x^2}\ dx =$$

Solution:

Given $\int \sqrt{3+8x-3x^2}\ dx$

$$3+8x-3x^2 = (-3)\left[x^2-\frac{8}{3}x-1\right]$$

$$= (-3)\left[\left(x-\frac{4}{3}\right)^2-\frac{16}{9}-1\right]$$

$$= (-3)\left[\left(x-\frac{4}{3}\right)^2-\frac{25}{9}\right]$$

$$= 3\left[\left(\frac{5}{3}\right)^2-\left(x-\frac{4}{3}\right)^2\right]$$

$$\therefore \int \sqrt{3+8x-3x^2}\ dx = \sqrt{3}\int \sqrt{\left[\left(\frac{5}{3}\right)^2-\left(x-\frac{4}{3}\right)^2\right]}dx$$

Put $x-\frac{4}{3}=t \Rightarrow dx = dt$

$$= \sqrt{3}\int \sqrt{\left[\left(\frac{5}{3}\right)^2-t^2\right]}\,dt$$

$$= \sqrt{3}\left[\frac{t\sqrt{\left[\left(\frac{5}{3}\right)^2-t^2\right]}}{2}+\frac{\left(\frac{5}{3}\right)^2}{2}\sin^{-1}\left(\frac{t}{\frac{5}{3}}\right)\right]+C \quad (\because \textit{formula}\ 29)$$

$$= \sqrt{3}\left[\frac{\left(x-\frac{4}{3}\right)\sqrt{\left[\left(\frac{5}{3}\right)^2-\left(x-\frac{4}{3}\right)^2\right]}}{2}+\frac{25}{18}\sin^{-1}\left(\frac{\left(x-\frac{4}{3}\right)}{\frac{5}{3}}\right)\right]+C$$

$$= \left[\frac{\left(x-\frac{4}{3}\right)\sqrt{3+8x-3x^2}}{2}+\frac{25}{6\sqrt{3}}\sin^{-1}\left(\frac{3x-4}{5}\right)\right]+C$$

$$= \left[\frac{(3x-4)\sqrt{3+8x-3x^2}}{6}+\frac{25}{6\sqrt{3}}\sin^{-1}\left(\frac{3x-4}{5}\right)\right]+C$$

Hence,

$$\int \sqrt{3+8x-3x^2}\ dx = \left[\frac{(3x-4)\sqrt{3+8x-3x^2}}{6}+\frac{25}{6\sqrt{3}}\sin^{-1}\left(\frac{3x-4}{5}\right)\right]+C$$

IP2.

Prove that

$$\int\sqrt{a^2+x^2}\,dx=\frac{x}{2}\sqrt{a^2+x^2}+\frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right)+C$$

where C is an arbitrary constant.

Proof:

Step1:

To evaluate $\int\sqrt{a^2+x^2}\,dx$,

Put $x=a\,sinh\theta,\ \theta\in\mathbb{R}$

$\Longrightarrow dx=a\,cosh\theta\,d\theta$

Step2:

$$\therefore\int\sqrt{a^2+x^2}\,dx=\int\sqrt{a^2+(a\,sinh\theta)^2}\ .a\,cosh\theta d\theta$$

$$=a\int\sqrt{1+sinh^2\theta}\,.a\,cosh\theta d\theta$$

$$=a^2\int cosh^2\theta\,d\theta$$

$$=a^2\int\left(\frac{1+cosh2\theta}{2}\right)d\theta$$

$$=\frac{a^2}{2}\left[\int d\theta+\int cosh2\theta\,d\theta\right]$$

$$=\frac{a^2}{2}\left[\theta+\frac{sinh2\theta}{2}\right]+C$$

where C is an arbitrary constant.

$$=\frac{a^2}{2}[\theta+sinh\theta.cosh\theta]+C\ldots\ldots\ldots\ (1)$$

Step3:

We have $sinh\theta=\frac{x}{a}\Longrightarrow\theta=\sinh^{-1}\left(\frac{x}{a}\right)$

$$(1)\Longrightarrow\int\sqrt{a^2+x^2}\,dx$$

$$=\frac{a^2}{2}\left[\theta+sinh\theta.\sqrt{1+sinh^2\theta}\right]+C$$

$$=\frac{a^2}{2}\left[\sinh^{-1}\left(\frac{x}{a}\right)+\frac{x}{a}\sqrt{1+\left(\frac{x}{a}\right)^2}\right]+C$$

$$=\frac{a^2}{2}\left[\sinh^{-1}\left(\frac{x}{a}\right)+\frac{x}{a^2}\sqrt{a^2+x^2}\right]+C$$

$$=\frac{x}{2}\sqrt{a^2+x^2}+\frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right)+C$$

P2.

$$\int \sqrt{3x^2 + 2x + 1}\ dx =$$

Solution:

Given $\int \sqrt{3x^2+2x+1}\ dx$

$$3x^2+2x+1 = 3\left[x^2+\tfrac{2}{3}x+\tfrac{1}{3}\right]$$

$$= 3\left[\left(x+\tfrac{1}{3}\right)^2+\tfrac{1}{3}-\tfrac{1}{9}\right]$$

$$= 3\left[\left(x+\tfrac{1}{3}\right)^2+\left(\tfrac{\sqrt{2}}{3}\right)^2\right]$$

$$\therefore \int \sqrt{3x^2+2x+1}\ dx = \sqrt{3}\int \sqrt{\left[\left(x+\frac{1}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2\right]}\ dx$$

Put $x+\frac{1}{3}=t \Rightarrow dx = dt$

$$= \sqrt{3}\int \sqrt{\left[\left(\tfrac{\sqrt{2}}{3}\right)^2+t^2\right]}\ dx$$

$$= \sqrt{3}\left[\frac{t}{2}\sqrt{\left[\left(\tfrac{\sqrt{2}}{3}\right)^2+t^2\right]}+\frac{\left(\frac{\sqrt{2}}{3}\right)^2}{2}\ln\left(t+\sqrt{\left[\left(\tfrac{\sqrt{2}}{3}\right)^2+t^2\right]}\right)\right]$$

$(\because\ \ \boldsymbol{formula\ 21})$

$$= \sqrt{3}\left[\frac{\left(x+\frac{1}{3}\right)}{2}\sqrt{\left[\left(\tfrac{\sqrt{2}}{3}\right)^2+\left(x+\tfrac{1}{3}\right)^2\right]}+\frac{4}{18}\ln\left(\left(x+\tfrac{1}{3}\right)+\sqrt{\left[\left(\tfrac{\sqrt{2}}{3}\right)^2+\left(x+\tfrac{1}{3}\right)^2\right]}\right)\right]$$

$$= \frac{3x+1}{6}\sqrt{3x^2+2x+1}+\frac{4}{6\sqrt{3}}\ln\left(\left(x+\frac{1}{3}\right)+\frac{1}{\sqrt{3}}\sqrt{3x^2+2x+1}\right)$$

Hence,

$\int \sqrt{3x^2+2x+1}\ dx$

$$= \frac{3x+1}{6}\sqrt{3x^2+2x+1}+\frac{4}{6\sqrt{3}}\ln\left(\left(x+\frac{1}{3}\right)+\frac{1}{\sqrt{3}}\sqrt{3x^2+2x+1}\right)$$

IP3.

$$\int \boldsymbol{x^2 \tan^{-1} 2x \; dx} =$$

Solution:

Step1:

$$\int x^2 \tan^{-1} 2x \; dx$$

$$= \frac{x^{2+1}}{2+1} \tan^{-1} 2x - \frac{2}{2+1} \int \frac{x^{2+1}}{1+4x^2} \; dx \quad (\boldsymbol{Formula\ 101})$$

$$= \frac{x^3}{3} \tan^{-1} 2x - \frac{2}{3} \int \frac{x^3}{1+4x^2} \; dx$$

Step2:

Put $4x^2 = t \Rightarrow x^2 = \frac{t}{4} \Rightarrow 2x\,dx = \frac{1}{4}\,dt \Rightarrow x\,dx = \frac{1}{8}\,dt$

$$\int x^2 \tan^{-1} 2x \; dx$$

$$= \frac{x^3}{3} \tan^{-1} 2x - \frac{2}{3} . \frac{1}{8} \int \frac{\frac{t}{4}}{1+t} \; dt$$

$$= \frac{x^3}{3} \tan^{-1} 2x - \frac{1}{48} \int \frac{t+1-1}{t+1} \; dt$$

$$= \frac{x^3}{3} \tan^{-1} 2x - \frac{1}{48} \int \frac{t+1}{t+1} \; dt - \frac{1}{48} \int \frac{1}{t+1} \; dt$$

$$= \frac{x^3}{3} \tan^{-1} 2x - \frac{t}{48} - \frac{1}{48} \log(t+1) + C$$

$$= \frac{x^3}{3} \tan^{-1} 2x - \frac{4x^2}{48} - \frac{1}{48} \log(4x^2+1) + C$$

$$= \frac{x^3}{3} \tan^{-1} 2x - \frac{x^2}{12} - \frac{1}{48} \log(4x^2+1) + C$$

Step3:

Hence,

$$\int x^2 \tan^{-1} 2x \; dx = \frac{x^3}{3} \tan^{-1} 2x - \frac{x^2}{12} - \frac{1}{48} \log(4x^2+1) + C$$

P3.

$$\int x^3 \tan^{-1} x \; dx =$$

Solution:

$$\int x^3 \tan^{-1} x \; dx$$

$$= \frac{x^{3+1}}{3+1} \tan^{-1} x - \frac{1}{3+1} \int \frac{x^{3+1}}{1+x^2} \; dx$$

$$(\boldsymbol{Formula\ 101})$$

$$= \frac{x^4}{4} \tan^{-1} x - \frac{1}{4} \int \frac{x^4}{1+x^2} \; dx$$

$$= \frac{x^4}{4} \tan^{-1} x - \frac{1}{4} \int \frac{x^4 - 1 + 1}{1+x^2} \; dx$$

$$= \frac{x^4}{4} \tan^{-1} x - \frac{1}{4} \int \frac{x^4 - 1}{1+x^2} \; dx - \frac{1}{4} \int \frac{1}{1+x^2} \; dx$$

$$= \frac{x^4}{4} \tan^{-1} x - \frac{1}{4} \int (x^2 - 1) \; dx - \frac{1}{4} \tan^{-1} x$$

$$(\boldsymbol{Formula\ 16})$$

$$= \frac{x^4}{4} \tan^{-1} x - \frac{1}{4} \int x^2 \; dx + \frac{1}{4} \int dx - \frac{1}{4} \tan^{-1} x$$

$$= \frac{x^4}{4} \tan^{-1} x - \frac{x^3}{12} + \frac{x}{4} - \frac{1}{4} \tan^{-1} x + C$$

$$= \left(\frac{x^4 - 1}{4}\right) \tan^{-1} x - \frac{x^3}{12} + \frac{x}{4} + C$$

Hence, $\int x^3 \tan^{-1} x \; dx = \left(\frac{x^4 - 1}{4}\right) \tan^{-1} x - \frac{x^3}{12} + \frac{x}{4} + C$

IP4.

$$\int \boldsymbol{x^3 cosh4x\ dx} =$$

Solution:

Step1:

$$\int x^3 cosh4x\ dx$$

$$= \frac{x^3}{4} sinh4x - \frac{3}{4}\int x^{3-1} sinh4x\, dx \quad (\textbf{\textit{Formula}}\ \mathbf{122})$$

$$= \frac{x^3}{4} sinh4x - \frac{3}{4}\int x^2 sinh4x\, dx$$

$$= \frac{x^3}{4} sinh4x - \frac{3}{4}\left[\frac{x^2}{4} cosh4x - \frac{2}{4}\int x^{2-1} cosh4x\, dx\right]$$

$$(\textbf{\textit{Formula}}\ \mathbf{121})$$

$$= \frac{x^3}{4} sinh4x - \frac{3x^2}{16} cosh4x + \frac{3}{8}\int x cosh4x\, dx$$

$$= \frac{x^3}{4} sinh4x - \frac{3x^2}{16} cosh4x + \frac{3}{8}\left[\frac{x}{4} sinh4x - \frac{1}{4}\int sinh4x\ dx\right]$$

$$(\textbf{\textit{Formula}}\ \mathbf{121})$$

$$= \frac{x^3}{4} sinh4x - \frac{3x^2}{16} cosh4x + \frac{3x}{32} sinh4x - \frac{3}{32}.\frac{1}{4} cosh4x + C$$

$$(\textbf{\textit{Formula}}\ \mathbf{113})$$

$$= \frac{x^3}{4} sinh4x - \frac{3x^2}{16} cosh4x + \frac{3x}{32} sinh4x - \frac{3}{128} cosh4x + C$$

$$= \left(\frac{x^3}{4} + \frac{3x}{32}\right) sinh4x - \left(\frac{3x^2}{16} + \frac{3}{128}\right) cosh4x + C$$

Step2:

Hence,

$$\int x^3 cosh4x\ dx = \left(\frac{x^3}{4} + \frac{3x}{32}\right) sinh4x - \left(\frac{3x^2}{16} + \frac{3}{128}\right) cosh4x + C$$

P4.

$$\int x^3 \sinh 3x \, dx =$$

Solution:

$$\int x^3 sinh3x\ dx$$

$$= \frac{x^3}{3} cosh3x - \frac{3}{3}\int x^{3-1} cosh3x\, dx \quad (\boldsymbol{Formula\ 121})$$

$$= \frac{x^3}{3} cosh3x - \int x^2 cosh3x\, dx$$

$$= \frac{x^3}{3} cosh3x - \left[\frac{x^2}{3} sinh3x - \frac{2}{3}\int x^{2-1} sinh3x\, dx\right]$$

$$(\boldsymbol{Formula\ 122})$$

$$= \frac{x^3}{3} cosh3x - \frac{x^2}{3} sinh3x + \frac{2}{3}\int x sinh3x\, dx$$

$$= \frac{x^3}{3} cosh3x - \frac{x^2}{3} sinh3x + \frac{2}{3}\left[\frac{x}{3} cosh3x - \frac{1}{3}\int cosh3x\ dx\right]$$

$$(\boldsymbol{Formula\ 121})$$

$$= \frac{x^3}{3} cosh3x - \frac{x^2}{3} sinh3x + \frac{2x}{9} cosh3x - \frac{2}{9}.\frac{1}{3} sinh3x + C$$

$$(\boldsymbol{Formula\ 114})$$

$$= \frac{x^3}{3} cosh3x - \frac{x^2}{3} sinh3x + \frac{2x}{9} cosh3x - \frac{2}{27} sinh3x + C$$

$$= \left(\frac{x^3}{3} + \frac{2x}{9}\right) cosh3x - \left(\frac{x^2}{3} + \frac{2}{27}\right) sinh3x + C$$

Hence,

$$\int x^3 sinh3x\ dx = \left(\frac{x^3}{3} + \frac{2x}{9}\right) cosh3x - \left(\frac{x^2}{3} + \frac{2}{27}\right) sinh3x + C$$

HWA-24(11.5&11.6)

Answer all the questions and submit

11.5

1. Find the following integrals:

a. $\int \frac{1+x^2}{\sqrt{1-x^2}}\,dx$

b. $\int \frac{\sqrt{y^2-4}}{y^3}\,dy$

c. $\int \frac{1}{2\sin x-\cos x+3}\,dx$

11.6

2. Find the following integrals:

a. $\int \sqrt{x^2+x+1}\,dx$

b. $\int e^x\sqrt{e^{2x}+1}\,dx$

c. $\int x\cot^{-1}x\,dx$

d. $\int \sqrt{7x-10-x^2}\,dx$

e. $\int x^2\sinh 2x\,dx$

1. $\int \tan^{-1} \sqrt{\frac{1-x}{1+x}}\, dx$

2. $\int \tan^{-1}\left(\frac{2x}{1+x^2}\right) dx$

3. $\int \frac{dx}{\sqrt{4-x^2}}$

4. $\int \sqrt{9x^2 - 25}\, dx$

5. $\int \frac{dx}{1+4x^2}$

6. $\int \sqrt{16 - 25x^2}\, dx$

7. $\int \frac{dx}{(1+x^2)\sqrt{1-x^2}}$

8. $\int \frac{dx}{4x^2 - 4x - 7}$

9. $\int \frac{ds}{(9-s^2)^2}$

10. $\int x^2 \sin^{-1} x \, dx$

11. $\int e^{2t} cos3t\ dt$

12. $\int sinh^5 2x\, dx$

13. $\int cosh^5 3x\, dx$

14. $$\int \frac{\tan^{-1} x}{(1+x)^2}$$

15. $\int \sqrt{1 + 3x - x^2}\, dx$